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A CLOSED SIMPLICIAL MODEL CATEGORY FOR PROPER HOMOTOPY AND SHAPE THEORIES

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In this paper, we introduce the notion of exterior space and give a full embedding of the category **P** of spaces and proper maps into the category **E** of exterior spaces. We show that the category **E** admits the structure of a closed simplicial model category. This technique solves the problem of using homotopy constructions available in the localised category **HoE** and in the "homotopy category" $\pi_0\mathbf{E}$, which can not be developed in the proper homotopy category.

On the other hand, for compact metrisable spaces we have formulated sets of shape morphisms, discrete shape morphisms and strong shape morphisms in terms of sets of exterior homotopy classes and for the case of finite covering dimension in terms of homomorphism sets in the localised category.

As applications, we give a new version of the Whitehead Theorem for proper homotopy and an exact sequence that generalises Quigley's exact sequence and contains the shape version of Edwards-Hastings' Comparison Theorem.

1. Introduction

One of the main applications of proper homotopy theory is the study of non compact spaces. For example, the classification of non compact surfaces given by Kérékjárto in 1923 used the notion of ideal point that can be considered as the first invariant of proper homotopy theory. Freudenthal [**11**] generalised this notion introducing the end point of a space and the end of a group. We can also cite the works of Siebenmann; in his thesis [**22**] he analysed the obstruction to finding a boundary to an open manifold in dimension greater than five, in [**23**] he also proved important s-cobordism theorems. The proof of the Poincare conjecture in dimension four was given by Fredmann [**10**] by using s-cobordism theorems and techniques of proper homotopy theory. We also want to mention the relationship between proper homotopy theory and shape theories, for this subject we refer the reader to [**9**]. For an interesting survey of the algebraic aspects of proper homotopy theory we refer the reader to [**18**]. We can summarise by saying that

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there are important applications of proper homotopy theory to the study of manifolds, ends of a group, shape and prohomotopy theories, et cetera. All these reasons and in particular the study of non compact manifolds have motivated the authors to develop new techniques in order to study proper homotopy invariants. The objective of our paper is to establish a nice framework for the study of proper homotopy theory.

One of the main problems of the proper category is that there are few limits and colimits. For this reason, we are not able to develop some homotopy constructions as homotopy fibres or loop spaces. In order to establish a framework for proper homotopy theory one can use homotopy theories with few categorical properties. Using this method one can study the homotopy constructions that can be developed inside the proper homotopy category. For instance, using the notion of cofibred category introduced by Baues [**2**], one can show that the proper category has the structure of a cofibred category, see [**5, 1, 3**].

Nevertheless, there are other possibilities, for example, we can embed the proper category into a complete and cocomplete category and use homotopy theories that assume the existence of limits and colimits. The last technique has the advantage that you can construct the analogues of the standard homotopy constructions such as homotopy fibre, loop spaces, et cetera. In this direction, one has the Edwards-Hastings embedding [**9**] of the proper homotopy category of locally compact σ-compact Hausdorff spaces into the homotopy category of pro-spaces and some results of Porter [**17**]. One disadvantage of this embedding is the restriction to locally compact σ-compact Hausdorff spaces. Another problem of this technique is that the homotopy constructions produce prospaces that many times can not be geometrically interpreted as a space.

In our paper we propose a new solution: a notion of exterior space is introduced in such a way that the category of exterior spaces is complete and cocomplete and we show that the proper category can be considered as a full subcategory of the category of exterior spaces. One of the main results of our paper is Theorem 4.1 in which we show that the category $\mathbf{E}$ of exterior spaces admits the structure of a closed simplicial model category. Therefore the results and properties developed by Quillen [**20, 21**] can be applied to the homotopy theory of exterior spaces. The closed model categories have been very useful to study and give algebraic models of rational homotopy theory. Recently, these Quillen's models have been used successfully to study some localisation and colocalisation functors, see [**8, 13**].

For the closed simplicial model structure of the category $\mathbf{E}$ of exterior spaces, we can consider the localised category $\mathbf{Ho}(\mathbf{E})$ obtained by inverting weak equivalences and the "homotopy category" $\pi_0\mathbf{E}$ obtained dividing by exterior homotopy relations. The category $\mathbf{Ho}(\mathbf{E})$ is equivalent to the full subcategory of cofibrant spaces $\pi_0\mathbf{E}_{\mathbf{cof}}$. This implies a Whitehead Theorem for cofibrant objects and from this fact we have obtained a new version of the Whitehead Theorem for the proper category which is given in

Theorem 5.3.

On the other hand, we also have nice applications to shape and to strong shape theory. We show in Theorem 5.4 that for compact metrisable spaces with finite covering dimension the set of strong shape morphisms can be given as a homomorphism set in the localised category **Ho(E)**. For compact metrisable spaces we also see that the sets of strong shape morphisms, discrete shape morphisms and shape morphisms can be expressed as suitable sets of exterior homotopy classes. Moreover, for a pointed compact metrisable space Y, using the exponential laws given in section 3, we give a (Serre) fibration $Q_Y^{\mathbb{R}_+} \to Q_Y^{\mathbb{N}}$ of pointed spaces such that the induced long exact sequence given in Theorem 5.5 generalises Quigley's exact sequence and contains a shape version of the Comparison Theorem of Edwards-Hastings. The fibration $Q_Y^{\mathbb{R}_+} \to Q_Y^{\mathbb{N}}$ permits us to give sets of strong shape, discrete shape and shape morphisms as sets of standard homotopy classes. For a exterior space X, the topological space $X^{\mathbb{N}}$ contains homotopy information of the (augmented) prospace of exterior open subsets of X and the space $X^{\mathbb{R}_+}$ corresponds with the homotopy limit of this prospace.

2. Preliminaries

In this section we recall some of the notions and results that will be used in this paper. We begin by recalling the notion of closed model category introduced by Quillen [**20**].

Definition 2.1: A closed model category is a category **C** endowed with three distinguished classes of morphisms called cofibrations, fibrations and weak equivalences, satisfying the following properties:

CM1: **C** is closed under finite limits and colimits.

CM2: For two morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

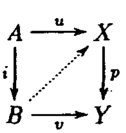
if any two of f, g and gf is a weak equivalence so is the third.

We say a morphism f in **C** is a retract of g if there are morphisms $\varphi : f \to g$ and $\psi : g \to f$ in the category of maps in **C**, such that $\psi\varphi = id$.

A morphism which is both fibration (respectively, cofibration) and weak equivalence is said to be a trivial fibration (respectively trivial cofibration).

CM3: If f is a retract of g, and g is a cofibration, fibration or weak equivalence, then so is f.

CM4: Given a commutative diagram of solid arrows:



(*)

the dotted arrow exist and the triangles commute, in either of the following cases:

(i) i is a cofibration and p is a trivial fibration,

(ii) i is a trivial cofibration and p is a fibration.

CM5: Any morphism f may be factored in two ways:

(i) $f = pi$, where i is a cofibration and p is a trivial fibration,

(ii) $f = qj$, where j is a trivial cofibration and q is a fibration.

If the dotted arrow exists in any diagram of the form (∗), in an arbitrary category, then we say that $i : A \to B$ has the left lifting property (LLP) with respect to $p : X \to Y$, and p has the right lifting property (RLP) with respect to i.

The initial object of $\mathbf{C}$ is denoted by $\emptyset$, and the final object by $*$. We call an object X of $\mathbf{C}$ cofibrant, if the unique morphism $\emptyset \to X$ is a cofibration; dually X is called fibrant if $X \to *$ is a fibration. We denote by $\mathbf{C_{cof}}$ and $\mathbf{C_{fib}}$ the full subcategories of $\mathbf{C}$ determined by cofibrant objects and fibrant objects, respectively.

The category $\mathbf{C}$ is said to be pointed if the initial and final objects are isomorphic. This object is usually denoted by $*$ and it is called the zero object.

We denote by $\mathbf{SS}$ the category of simplicial sets. It is known that $\mathbf{SS}$ is a closed model category. Quillen [**20**] gave the following structure: a simplicial map $f : X \to Y$ is said to be a fibration (respectively, trivial fibration) if it has the RLP with respect to $V(n,k) \hookrightarrow \Delta[n]$, for $0 \leqslant k \leqslant n$ and $n > 0$ (respectively, to $\dot{\Delta}[n] \hookrightarrow \Delta[n]$, for $n \geqslant 0$), where $V(n,k)$ is the simplicial subset generated by the i-faces, $i \neq k$, of the standard n-simplex $\Delta[n]$; $\dot{\Delta}[n]$ is generated by all the faces of $\Delta[n]$. A simplicial map $i : A \to B$ is said to be a cofibration (respectively, trivial fibration) if it has the LLP with respect to trivial fibrations (respectively, fibrations). A simplicial map f is a weak equivalence if it can be factored as $f = pi$ where i is a trivial cofibration and p is a trivial fibration.

Definition 2.2: A simplicial category is a category $\mathbf{C}$ endowed with a functor $\underline{Hom}_C : \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{SS}$, simplicial maps $\circ : \underline{Hom}_C(X,Y) \times \underline{Hom}_C(Y,Z) \to \underline{Hom}_C(X,Z)$, called composition, defined for each triple X, Y, Z of objects of $\mathbf{C}$, and a natural isomorphism $Hom_C(X,Y) \cong \underline{Hom}_C(X,Y)_0$, $(f \rightsquigarrow \tilde{f})$, satisfying the following:

(i) the composition map is associative,

(ii) if $u \in Hom_C(X,Y)$ and $f \in \underline{Hom}_C(Y,Z)_n$ then $\underline{Hom}_C(u,Z)_n(f) = f \circ s_0^n(\tilde{u})$. Also $\underline{Hom}_C(W,u)_n(g) = s_0^n(\tilde{u}) \circ g$ if $g \in \underline{Hom}_C(W,X)_n$.

Let X be an object of $\mathbf{C}$, and K a simplicial set. By $X \otimes K$ we mean an object of $\mathbf{C}$ with a simplicial map $\alpha : K \to \underline{Hom}_C(X, X \otimes K)$ which induces, in a natural way, see [**20**], an isomorphism $\underline{Hom}_C(X \otimes K, Y) \cong (\underline{Hom}_C(X,Y))^K$, for all objects Y of $\mathbf{C}$. Dually, X^K is an object of $\mathbf{C}$ with a simplicial map $\beta : K \to \underline{Hom}_C\left(X^K, X\right)$ which induces an isomorphism $\underline{Hom}_C\left(Y, X^K\right) \cong (\underline{Hom}_C(Y,X))^K$ for every object Y of C.

Definition 2.3: A closed simplicial model category is a closed model category $\mathbf{C}$ which is also a simplicial category and satisfies the following axioms:

SM0: If X is an object of $\mathbf{C}$ and K is a finite simplicial set then $X \otimes K$ and X^K exist.

SM7: If $i : A \to B$ is a cofibration and $p : X \to Y$ is a fibration then, the pull-back map:

$$\big(\underline{Hom}_C(B,p), \underline{Hom}_C(i,X)\big) : \underline{Hom}_C(B,X) \to \underline{Hom}_C(A,X) \times_{\underline{Hom}_C(A,Y)} \underline{Hom}_C(B,Y)$$

is a fibration of simplicial sets, which is trivial if either i or p is trivial.

In this paper in order to prove SM7, we shall use the following result given in [**20**].

Proposition 2.1. *Let* $\mathbf{C}$ *be a closed model category, which is a simplicial category satisfying* SM0. *Then* SM7 *is equivalent to:*

SM7(a): *If* $p : X \to Y$ *is a fibration in* $\mathbf{C}$ *(respectively, trivial fibration) then the pull-back morphism* $X^{\Delta[n]} \to X^{\dot\Delta[n]} \times_{Y^{\dot\Delta[n]}} Y^{\Delta[n]}$ *is a fibration (respectively, trivial fibration) and* $X^{\Delta[1]} \to X^{V(1,k)} \times_{Y^{V(1,k)}} Y^{\Delta[1]}$ *is a trivial fibration, for* $k \in \{0,1\}$.

Given a closed model category $\mathbf{C}$, the category of fractions obtained by formal inversion of the weak equivalences, see [**12**], is denoted by $\mathbf{Ho}(\mathbf{C})$ and the quotient category obtained by dividing by homotopy relations will be denoted by $\pi_0\mathbf{C}$, we note that $\pi_0\mathbf{C}(X,Y) = \pi_0\underline{Hom}_C(X,Y)$.

3. The category of exterior spaces

Recall that a continuous map $f : X \to Y$ is said to be proper if for every closed compact subset K of Y, $f^{-1}(K)$ is a compact subset of X. The category of spaces and proper maps will be denoted by $\mathbf{P}$. This category and the corresponding proper homotopy category are very useful for the study of non compact spaces. Nevertheless, one has the problem that this category does not have enough limits and colimits and then we can not develop the usual homotopy constructions like loops, homotopy limits and colimits, et cetera.

In this section, we give an answer to this problem introducing the notion of exterior space. The new category of exterior spaces and maps is complete and cocomplete and contains as a full subcategory the category of spaces and proper maps.

Definition 3.1: Let (X,τ) be a space. An externology on (X,τ) is a non empty collection ε of open subsets satisfying:

E1: If E_1, $E_2 \in \varepsilon$ then $E_1 \cap E_2 \in \varepsilon$,

E2: if $E \in \varepsilon$, $U \in \tau$ and $E \subset U$ then $U \in \varepsilon$.

An exterior space (or exterior topological space) $(X, \varepsilon \subset \tau)$ consists of a space (X,τ) together with an externology ε. An open E which is in ε is said to be an exterior-open subset or for shorting an e-open subset. A map $f : (X, \varepsilon \subset \tau) \to (X', \varepsilon' \subset \tau')$ is said to be exterior if it is continuous and $f^{-1}(E) \in \varepsilon$, for all $E \in \varepsilon'$. The category of exterior spaces and maps will be denoted by $\mathbf{E}$.

EXAMPLES.

(1) For a space (X,τ), one can always consider the trivial externology $\varepsilon = \{X\}$ and on the other hand the externology $\varepsilon = \tau$. Note that an externology ε is a topology if and only if the empty set $\emptyset$ is a member of ε if and only if $\varepsilon = \tau$.

(2) Given an space (X,τ), we always have the externology ε^X_{cc} of the complements of closed-compact subsets of X.

(3) If Y is a closed subspace of the Hilbert cube Q, we can consider the externology of those open subsets E such that $Y \subset E \subset Q$.

THEOREM 3.1. *There is a full embedding* $e : \mathbf{P} \hookrightarrow \mathbf{E}$.

PROOF: The functor e carries a space X to the exterior space X_e which is provided with the topology of X and the externology ε^X_{cc}. A map $f\colon X \to Y$ is carried to the exterior map $f_e\colon X_e \to Y_e$ given by $f_e = f$. It is easy to check that a map $f : X \to Y$ in $\mathbf{T}$ is proper if and only if $f = f_e\colon X_e \to Y_e$ is exterior. ∎

PROPOSITION 3.1. *The category* $\mathbf{E}$ *is complete and cocomplete.*

PROOF: It suffices to prove that $\mathbf{E}$ has all equalisers and products (respectively, all coequalisers and coproducts.)

Given $f, g : X \to Y$ in $\mathbf{E}$, we consider $A = \{x : f(x) = g(x)\} \subset X$ with the relative topology and the externology ε_A given by subsets of the form $E = E' \cap A$, $E' \in \varepsilon_X$. Then the inclusion map $i : A \hookrightarrow X$ is the equaliser of f and g. We also consider the quotient space $Y/\!\sim$, obtained by the equivalence relation generated by the relations $f(y) \sim g(y)$, $y \in Y$, provided with the externology $\varepsilon_{Y/\sim}$ of those $E \subset Y/\!\sim$ such that $\pi^{-1}(E) \in \varepsilon_Y$, where $\pi : Y \to Y/\!\sim$ is the quotient map; it is clear that π is the coequaliser of f and g. On the other hand, given a collection of exterior spaces $\{X_k\}_{k\in I}$ we consider the spaces $\prod_{k\in I} X_k$ and $\coprod_{k\in I} X_k$. If we denote by $p_\alpha : \prod_{k\in I} \to X_\alpha$ the α-th projection map, then the product externology for $\prod_{k\in I} X_k$ is given by all open sets such that contain a finite intersection of subsets of the form $p_j^{-1}(E_j)$ where $E_j \in \varepsilon_j$. We also have the sum externology for $\coprod_{k\in I} X_k$ given by the family of subsets E such that $E \cap X_k \in \varepsilon_k$, for all $k \in I$. It is easy to check that $\prod_{k\in I} X_k$ and $\coprod_{k\in I} X_k$ are the product and coproduct of $\{X_k\}_{k\in I}$ in $\mathbf{E}$, respectively. ∎

REMARK. $\mathbf{E}$ has an initial object $(\emptyset, \{\emptyset\} \subset \{\emptyset\})$ and a final object $(*, \{*\} \subset \{\emptyset, *\})$. These objects are not isomorphic so $\mathbf{E}$ is not a pointed category.

DEFINITION 3.2: Let X be an exterior space and $L \subset X$, we say that L is *e-compact* if $L \setminus E$ is a compact subset, for all E e-open subset of X.

Let X, Z be exterior spaces, then we define $Z^X = Hom_{\mathbf{E}}(X, Z)$ with the topology generated by the subsets of the form:

$$(K, U) = \{\alpha \in Z^X : \alpha(K) \subset U\}$$

$$(L,E) \;=\; \{\alpha \in Z^X : \alpha(L) \subset E\}$$

where $K \subset X$ is a compact subset, $U \subset Z$ is an open subset, $L \subset X$ is an e-compact subset and $E \subset Z$ an e-open subset. This construction gives a functor $\mathbf{E}^{op} \times \mathbf{E} \to \mathbf{T}$.

We note that if X is a Hausdorff, locally compact space and $\varepsilon_X = \varepsilon_{cc}^X$ it easy to show that the topology on Z^X is also generated by the subsets of the form (K,U) and $\left(\overline{E}_X, E_Z\right)$, where $E_X \in \varepsilon_X$, $E_Z \in \varepsilon_Z$ and $\overline{E}_X$ is the closure of E_X in X .

Let X be an exterior space, Y a topological space, we consider on $X \times Y$ the product topology and the externology given by those $E \in \varepsilon_{X\times Y}$ such that for each $y \in Y$ there exists $U_y \in \tau_Y$, $y \in U_y$ and $E_y \in \varepsilon_X$ such that $E_y \times U_y \subset E$. This exterior space will be denoted by $X \bar{\times} Y$ in order to avoid a possible confusion with the product externology. This construction gives a functor $\mathbf{E} \times \mathbf{T} \to \mathbf{E}$. When Y is a compact space we can prove that E is an e-open subset if and only if it is an open subset and there exists $G \in \varepsilon_X$ such that $G \times Y \subset E$. Furthermore, if Y is a compact space and $\varepsilon_X = \varepsilon_{cc}^X$ then $\varepsilon_{X \bar{\times} Y}$ coincides with the externology of complements of closed-compact subsets of $X \times Y$.

Let Y be a topological space and Z an exterior space, then we consider on $Z^Y = Hom_{\mathbf{T}}(Y,Z)$ the compact-open topology and the externology given by the open subsets E of Z^Y such that E contains a subset of the form (K,G), where K is a compact subset of Y and G is an e-open subset of Z. This construction gives a functor $\mathbf{T}^{op} \times \mathbf{E} \to \mathbf{E}$. It is not very difficult to see that, if Y is a compact space, $E \in \tau_{Z^Y}$ is an e-open subset if and only if it contains a subset of the form (Y,E).

THEOREM 3.2. *Let X, Z be exterior spaces and Y a topological space, then*

(i) If X is a Hausdorff, locally compact space and $\varepsilon_X = \varepsilon_{cc}^X$ there is a natural bijection

$$Hom_{\mathbf{E}}(X \bar{\times} Y, Z) \cong Hom_{\mathbf{T}}\left(Y, Z^X\right)$$

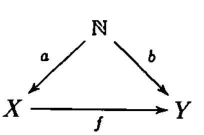
(ii) If Y is a locally compact space there is a natural bijection

$$Hom_{\mathbf{E}}(X \bar{\times} Y, Z) \cong Hom_{\mathbf{E}}\left(X, Z^Y\right)$$

PROOF: The proof is routine and is left as an exercise. ▯

In this paper we shall consider the exterior space $\mathbb{N}$ of non negative integers with the discrete topology and the externology $\varepsilon_{cc}^{\mathbf{N}}$. Note that $\mathbb{N}$ is a Hausdorff, locally compact space.

Let $\mathbf{E}^{\mathbf{N}}$ be the category of exterior spaces under $\mathbb{N}$. Recall that an object is given by a exterior map $a : \mathbb{N} \to X$, denoted by (X,a), and the morphisms are given by commutative triangles



denoted by $f : (X,a) \to (Y,b)$.

DEFINITION 3.3: Let f, g be in $Hom_{\mathbf{E}}^{\mathbb{N}}\big((X,a),(Y,b)\big)$, then we say f is e-homotopic to g relative to $\mathbb{N}$, written $f \simeq_e g$, if there is an exterior map $F : X \bar{\times} I \to Y$ such that $F(x,0) = f(x)$, $F(x,1) = g(x)$ and $F\big(a(k),t\big) = b(k)$, for all $x \in X$, $k \in \mathbb{N}$ and $t \in I$. The map F is called an exterior homotopy relative to $\mathbb{N}$ from f to g and we shall sometimes write $F : f \simeq_e g$. The set of exterior homotopy classes relative to $\mathbb{N}$ will be denoted by $\big[(X,a),(Y,b)\big]^{\mathbb{N}}$.

Let S^q denote the q-dimensional pointed sphere and let $\mathbb{N}\bar{\times}S^q$ be the exterior space obtained by the functor $\mathbf{E} \times \mathbf{T} \to \mathbf{E}$ described above. From the adjunction isomorphisms of Theorem 3.2:

$$\begin{aligned} Hom_{\mathbf{E}}(\mathbb{N}\bar{\times}S^q, X) &\cong Hom_{\mathbf{T}}\big(S^q, X^{\mathbb{N}}\big) \\ Hom_{\mathbf{E}}\big(\mathbb{N}\bar{\times}(S^q \times I), X\big) &\cong Hom_{\mathbf{T}}\big(S^q \times I, X^{\mathbb{N}}\big) \end{aligned}$$

one obtains the following result:

PROPOSITION 3.2. *Let (X,a) be in $\mathbf{E}^{\mathbb{N}}$. There is a natural isomorphism*

$$\big[(\mathbb{N}\bar{\times}S^q, id_{\mathbb{N}}\bar{\times}*), (X,a)\big]^{\mathbb{N}} \cong \big[(S^q,*)\big(X^{\mathbb{N}},a\big)\big]$$

where the second member is the standard set of pointed homotopy classes.

The canonical isomorphism of the proposition above induces on $\big[(\mathbb{N}\bar{\times}S^q, id_{\mathbb{N}}\bar{\times}*), (X,a)\big]^{\mathbb{N}}$ the structure of a group for $q \geqslant 1$ which is Abelian for $q \geqslant 2$. For $q = 0$ one has a pointed set.

DEFINITION 3.4: Let (X,a) be an object of $\mathbf{E}^{\mathbb{N}}$, for $q \geqslant 0$ the q-th exterior homotopy group functor of (X,a) is given by $\pi_q^e(X,a) = \big[(\mathbb{N}\bar{\times}S^q, id_{\mathbb{N}}\bar{\times}*), (X,a)\big]^{\mathbb{N}}$.

REMARK. Notice that $X^{\mathbb{N}}$ can be considered as a subset of $\prod_{\mathbb{N}} X$. If we take in $\prod_{\mathbb{N}} X$ the topology generated by $U_1 \times \ldots \times U_n \times E \times E \times \ldots$, where $U_1, \ldots, U_n$ are open subsets of X and E is an exterior open subset of X, we have a topology between the product topology and the box topology. We note that the relative topology is the topology given above on $X^{\mathbb{N}}$.

4. A CLOSED SIMPLICIAL MODEL CATEGORY STRUCTURE FOR $\mathbf{E}$

In this section we show that $\mathbf{E}$ has a closed simplicial model category structure with the following classes of maps:

DEFINITION 4.1: Let $f : X \to Y$ be an exterior map:

(i) f is a weak exterior equivalence, denoted by weak e-equivalence, in either of the following cases:

  (a) If $X^{\mathbb{N}} = \emptyset$ then $Y^{\mathbb{N}} = \emptyset$.

(b) If $X^{\mathbf{N}} \neq \emptyset$ then $\pi_q^e(f) : \pi_q^e((X,a)) \to \pi_q^e((Y,fa))$ is an isomorphism for all $a \in X^{\mathbf{N}}$, $q \geqslant 0$.

(ii) f is an exterior fibration or e-fibration if it has the RLP with respect to $\partial_0 : \mathbb{N}\bar{\times} D^q \to \mathbb{N}\bar{\times}(D^q \times I)$ for all $q \geqslant 0$, where $\partial_0(n,x) = (n,x,0)$.

A map which is both an e-fibration and a weak e-equivalence is said to be a trivial e-fibration.

(iii) f is an exterior cofibration or e-cofibration if it has the LLP with respect to any trivial e-fibration.

A map which is both an e-cofibration and a weak e-equivalence is said to be a trivial e-cofibration. An exterior space X is said to be e-fibrant or e-cofibrant, if $X \to *$ is an e-fibration or $\emptyset \to X$ is an e-cofibration, respectively.

In this paper we are using the simplicial closed model structure of the category $\mathbf{T}$ of spaces given by Quillen [**20**]. Given a map $f: X \to Y$ between exterior spaces, one has an induced continuous map $f^{\mathbf{N}}: X^{\mathbf{N}} \to Y^{\mathbf{N}}$. One can check that f is a weak e-equivalence if and only if $f^{\mathbf{N}}$ is a weak equivalence in $\mathbf{T}$ and f is an e-fibration if and only if $f^{\mathbf{N}}$ is a fibration in $\mathbf{T}$. Note that $X \to *$ is always an e-fibration so every object in $\mathbf{E}$ is e-fibrant.

Consider the exterior spaces $\mathfrak{S}^{n-1} = \mathbb{N}\bar{\times} S^{n-1}$, for $n = 0$ take $\mathfrak{S}^{-1} = \emptyset$, and $\mathfrak{D}^n = \mathbb{N}\bar{\times} D^n$.

Proposition 4.1. *Let $f : X \to Y$ be an exterior map, then f is a trivial e-fibration if and only if it has the RLP with respect to $\mathfrak{S}^{q-1} \hookrightarrow \mathfrak{D}^q$, for all $q \geqslant 0$.*

Proof: Taking into account the exponential law $Hom_{\mathbf{E}}(\mathbb{N}\bar{\times} Y, X) \cong Hom_{\mathbf{T}}\left(Y, X^{\mathbf{N}}\right)$, we have that f is a trivial e-fibration if and only if $f^{\mathbf{N}}$ is a trivial fibration in $\mathbf{T}$ or $f^{\mathbf{N}}$ has the RLP with respect to $S^{q-1} \hookrightarrow D^q$ for all $q \geqslant 0$. Applying again the same exponential law this is equivalent to say that f has the RLP with respect to $\mathfrak{S}^{q-1} \hookrightarrow \mathfrak{D}^q$, for all $q \geqslant 0$. ∎

This section is devoted to prove the following result which is one of the main theorems of our paper.

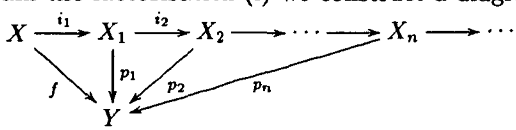
Theorem 4.1. *The category $\mathbf{E}$ of exterior spaces together with the classes of e-cofibrations, e-fibrations and weak e-equivalences has a closed simplicial model category structure.*

Proof: CM1 follows directly from proposition 3.1. Since f is a weak e-equivalence if and only if $f^{\mathbf{N}}$ is a weak equivalence in $\mathbf{T}$, we have CM2. On the other hand, since the notions of e-cofibration and e-fibration are defined by lifting properties, it is easy to check that the classes of e-fibrations and e-cofibrations are closed by retracts. Furthermore, a retract of an isomorphism is a isomorphism. Therefore CM3 is satisfied. We now prove CM5; for any morphism $f : X \to Y$ in $\mathbf{E}$ we have to prove that it can be factored in two ways:

(i) $f = pi$, where i is an e-cofibration and p is a trivial e-fibration.

(ii) $f = qj$, where j is a trivial e-cofibration and q is an e-fibration.

First, in order to obtain the factorisation (i) we construct a diagram:



Take $X_0 = X$, $p_0 = f$ and assume obtained X_{n-1}, consider the set Λ of commutative diagrams:

$$\begin{array}{ccc} \mathfrak{S}^{q_\lambda - 1} & \xrightarrow{u_\lambda} & X_{n-1} \\ \downarrow & & \downarrow p_{n-1} \\ \mathfrak{D}^{q_\lambda} & \xrightarrow[u_\lambda]{} & Y \end{array} \qquad (q_\lambda \geq 0)$$

Then define $i_n : X_{n-1} \to X_n$ by the following push-out:

$$\begin{array}{ccc} \coprod_{\lambda\in\Lambda} \mathfrak{S}^{q_\lambda - 1} & \xrightarrow{\coprod_{\lambda\in\Lambda} u_\lambda} & X_{n-1} \\ \downarrow & & \downarrow i_n \\ \coprod_{\lambda\in\Lambda} \mathfrak{D}^{q_\lambda - 1} & \xrightarrow[\coprod_{\lambda\in\Lambda} w_\lambda]{} & X_n \end{array} \qquad (*)$$

and $p_n : X_n \to Y$ is the sum of p_{n-1} and all maps v_λ, $\lambda \in \Lambda$, so p_n extends p_{n-1}. We take $\widetilde{X} = colim\ X_n$ and $p = colim\ p_n$.

If we denote the natural inclusion of X_n into $\widetilde{X}$ by $k_n : X_n \to \widetilde{X}$, and consider $i = k_0$, it follows that $f = pi$. In order to show that i is an e-cofibration observe that, by proposition 4.1, $\mathfrak{S}^{q_\lambda - 1} \hookrightarrow \mathfrak{D}^{q_\lambda}$ is an e-cofibration for all $q_\lambda \geqslant 0$, therefore $\coprod_{\lambda\in\Lambda} \mathfrak{S}^{q_\lambda - 1} \hookrightarrow \coprod_{\lambda\in\Lambda} \mathfrak{D}^{q_\lambda}$ is an e-cofibration too. That each i_n is an e-cofibration follows from the fact that it is the cobase extension of an e-cofibration. Now suppose that we have a commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{u} & E \\ i\downarrow & & \downarrow q \\ \widetilde{X} & \xrightarrow[v]{} & B \end{array}$$

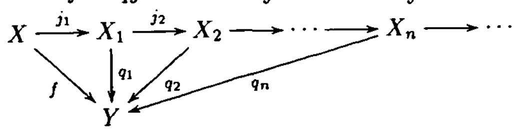
where q is a trivial e-fibration. Since i_1 is an e-cofibration, consider the lifting $l_1 : X_1 \to E$ such that $l_1 i_1 = u$ and $q l_1 = v j_1$. By an induction argument we take $l_n : X_n \to E$ such that $l_n i_n = l_{n-1}$ and $q l_n = v k_n$. Now we can consider $l = colim\ l_n : \widetilde{X} \to E$, it is easy to check that $li = u$ and $ql = v$.

We apply proposition 4.1 to show that $p : \widetilde{X} \to Y$ is a trivial fibration. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{S}^{q_\lambda - 1} & \xrightarrow{\alpha_\lambda} & \widetilde{X} \\ \downarrow & & \downarrow p \\ \mathfrak{D}^{q_\lambda - 1} & \xrightarrow[w_\lambda]{} & Y \end{array} \qquad (q_\lambda \geq 0)$$

We assume that $\alpha_\lambda : \mathfrak{S}^{q_\lambda - 1} \to \widetilde{X}$ factors through X_m for m sufficiently large, $\alpha_\lambda = k_m u_\lambda$, where $k_m : X_m \to \widetilde{X}$ is the natural inclusion. Taking into account the construction of X_{m+1} there exists $w_\lambda : \mathfrak{D}^{q_\lambda} \to X_{m+1}$ such that $p_{m+1} w_\lambda = v_\lambda$ and $w_\lambda|_{\mathfrak{S}^{q_\lambda - 1}} = i_{m+1} v_\lambda$. Then $h = k_{m+1} w_\lambda$ is a lifting for the diagram above.

The other factorisation $f = qj$ is similarly obtained by constructing a diagram:



Take $X_0 = X$, $q_0 = f$ and suppose X_{n-1} constructed, then consider the set Λ of commutative diagrams:

$$\begin{array}{ccc} \mathfrak{D}^{q_\lambda} & \xrightarrow{u_\lambda} & X_{n-1} \\ {\scriptstyle \partial_0^\lambda} \downarrow & & \downarrow {\scriptstyle q_{n-1}} \\ \mathfrak{D}^{q_\lambda} \bar{\times} I & \xrightarrow[v_\lambda]{} & Y \end{array} \qquad (q_\lambda \geq 0)$$

Then we obtain $j_n : X_{n-1} \to X_n$ as the push-out of $\coprod_{\lambda \in \Lambda} u_\lambda$ and $\coprod_{\lambda \in \Lambda} \partial_\lambda^0$; $q_n : X_n \to Y$ is the sum of q_{n-1} and all maps v_λ, $\lambda \in \Lambda$. Taking $\widetilde{X} = colim\ X_n$, $q = colim\ q_n$ and $j = X \to \widetilde{X}$ the natural inclusion, we have that $f = qj$. Using similar arguments to those used in factorisation (i), we can prove that j is an e-cofibration and q is an e-fibration. It is not very difficult to prove that $j_n : X_{n-1} \to X_n$ is a strong deformation retract since each ∂_0^λ is a strong deformation retract. Therefore one has bijections $\pi_q^e(j) : \pi_q^e((X, a)) \to \pi_q^e\big((\widetilde{X}, ia)\big)$ when $X^{\mathbb{N}} \neq \emptyset$, and using again the assumption that exterior maps of the form $\mathfrak{S}^q \to \widetilde{X}$, $\mathfrak{S}^q \bar{\times} I \to \widetilde{X}$ factor through X_m for m sufficiently large. Furthermore, j has the LLP with respect to e-fibrations. Thus CM5 is satisfied.

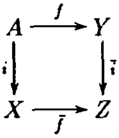
Clearly CM4 (i) is a consequence from the definition of e-cofibration. We derive CM4 (ii) from CM5 and CM2 as follows. Suppose that i is a trivial e-cofibration, then by CM5 (ii), it can be factored as $i = qj$, where $j : A \to Z$ is an e-cofibration having the LLP with respect to e-fibrations and q is an e-fibration. Since CM2 holds, q is a trivial e-fibration. Then there is a lifting $r : B \to Z$ in the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{j} & Z \\ {\scriptstyle i} \downarrow & & \downarrow {\scriptstyle q} \\ B & \xrightarrow[id_B]{} & Z \end{array}$$

So the map i is a retract of j. Therefore i has the LLP with respect to e-fibrations. ∎

In the proof of 4.1 we have used the assumption that an exterior map of the form $\mathbb{N} \bar{\times} K \to \widetilde{X}$, where K is a Hausdorff compact space, factors through X_m for m sufficiently large. We are going to prove this fact in proposition 4.2. Next we analyse some properties of the filtrations above. Observe that each exterior map $X_{n-1} \to X_n$ is injective because it is the cobase change of a injective map, so we can write $X \subset X_1 \subset X_2 \subset \cdots \subset X_n \subset \cdots \subset \widetilde{X}$. On the other hand, we can suppose that $X_n \setminus X_{n-1} \neq \emptyset$, for all n.

We use the fact that in a push-out diagram in $\mathbf{E}$ of the form:



if i is injective and closed (e-closed) then $\bar{\imath}$ is also a closed (e-closed) map. By the form in which X_n has been constructed, we infer that X_{n-1} is closed and e-closed in X_n. So X is closed and e-closed in $\widetilde{X}$ taking into account that $\widetilde{X}$ has the weak topology and the weak externology. Using these properties the filtrations used to factor maps one has:

Lemma 4.1.

(a) *$\{p\}$ is closed and e-closed in X_n, for all $p \in X_n \setminus X$.*

(b) *If $K \subset \widetilde{X}$, where K is a compact subset, then there is $n \in \mathbb{N}$ such that $K \subset X_n$.*

Proof: (a) If $p \in X_n \setminus X$, then there is m, $1 < m \leqslant n$, such that $p \in X_m \setminus X_{m-1}$, thus for the first kind of filtration $p \in \mathfrak{D}^{q_\lambda}$ or for the second kind of filtration $p \in \mathfrak{D}^{q_\lambda} \bar{\times} I$. Because we consider in both cases the externology of the complements of closed-compact subsets, we have that $\{p\}$ is e-closed. Furthermore, since $\{p\} \cap X_k = \emptyset$ for all $k \leqslant m-1$, we have that $\{p\}$ is closed and e-closed in X_m . We have that $X_{r-1} \to X_r$ is closed and e-closed so $X_m \to X_n$ is also closed and e-closed. Therefore we have that $\{p\}$ is closed and e-closed in X_n.

(b) Suppose that $K \not\subset X_n$ for all n. Then there is a sequence $\{k_i\}_{i\in\mathbb{N}} \subset K$ satisfying that $k_i \in X_{n_i} \setminus X_{n_i-1}$, $k_i \notin X$, for all $i \in \mathbb{N}$. Consider the subspace $T = \{k_i : i \in \mathbb{N}\}$. If $S \subset T$, then we have that $S \cap X_n$ is finite and $S \cap X_n \cap X = \emptyset$. Applying (a) we have that $S \cap X_n$ is closed for every n . Thus S is closed in $\widetilde{X}$, therefore T has the discrete topology and is closed in $\widetilde{X}$. Since $T \subset K$, it follows that T is compact; this contradicts the fact that T is an infinite discrete space. □

Proposition 4.2. *Let Z be an exterior space, Hausdorff, σ-compact, locally compact space and having the externology of the complements of closed-compact subsets. Then, given $f : Z \to \widetilde{X}$ an exterior map, f factors through X_n, for n sufficiently large.*

Proof: We consider an increasing sequence of compact subsets $\{K_n\}_{n\in\mathbb{N}}$ such that $K_n \subset Int(K_{n+1})$, $Z = \bigcup\limits_{n\in\mathbb{N}} K_n$, and suppose that f does not factor through any X_n. By Lemma 4.1 we have that $f(K_i) \subset X_{n_i}$, $n_i > n_{i-1}$. We can construct a sequence $a(i) \in Z$ such that $f\big(a(i)\big) \in X_{m_i} \setminus X_{m_i-1}$ and $a(1) \in X_{m_1} \setminus X$. Now we define a map $\sigma = fa : \mathbb{N} \to \widetilde{X}$; one can check that σ is an exterior map and satisfies $\sigma(i) \in X_{n_i} \setminus X_{n_i-1}$, $\sigma(1) \notin X$. On the other hand $Im\,\sigma \cap X_n$ is finite and therefore e-closed in X_n for all n, so $Im\ \sigma$ is e-closed in $\widetilde{X}$. Since $\sigma^{-1}(Im\ \sigma) = \mathbb{N}$ we obtain that $\mathbb{N}$ is e-closed contradicting the fact that $\mathbb{N}$ is not compact. □

We now define the functor $\underline{Hom}_{\mathbf{E}} : \mathbf{E}^{op} \times \mathbf{E} \to \mathbf{SS}$ by

$$\begin{aligned} \underline{Hom}_{\mathbf{E}}(X,Y)_n &= Hom_{\mathbf{E}}(X \bar{\times} |\Delta[n]|, Y) \\ \underline{Hom}_{\mathbf{E}}(X,Y)(\varphi) &= Hom_{\mathbf{E}}(id_X \bar{\times} |\varphi_*|, id_Y) \\ \underline{Hom}_{\mathbf{E}}(f,g)_n &= Hom_{\mathbf{E}}\left(f \bar{\times} \left|id_{\Delta[n]}\right|, g\right) \end{aligned}$$

Here we consider the category of simplicial sets as a functor category $\mathbf{SS} = \mathbf{Sets}^{\mathbf{\Delta^{op}}}$, where Δ is the category whose objects are all the finite ordered sets $[n] = \{0 < 1 < \cdots < n\}$ and whose morphisms $[n] \to [m]$ are those maps $\varphi : [n] \to [m]$ which preserve the order.

If $f \in \underline{Hom}_{\mathbf{E}}(X,Y)_n$ and $g \in \underline{Hom}_{\mathbf{E}}(Y,Z)_n$, let $g \circ_n f$ be the composite:

$$X \bar{\times} |\Delta[n]| \xrightarrow{id_X \bar{\times} \Delta} X \bar{\times} (|\Delta[n]| \times |\Delta[n]|) \xrightarrow{f \bar{\times} id_{|\Delta[n]|}} Y \bar{\times} |\Delta[n]| \xrightarrow{g} Z$$

where $\Delta = (id, id)$ is the diagonal map. We also have a natural isomorphism $Hom_{\mathbf{E}}(X,Y) \cong \underline{Hom}_{\mathbf{E}}(X,Y)_0$ since $|\Delta[0]| \cong *$. One can check that $\mathbf{E}$ with this structure is a simplicial category.

DEFINITION 4.2: Let X be an exterior space and K a finite simplicial set. We define $X \otimes K = X \bar{\times} |K|$ and $X^K = X^{|K|}$.

Observe that $|K|$ is a Hausdorff compact space if K is a finite simplicial set.

Given a small category $\mathbf{I}$, the functor category $\mathbf{Sets}^{\mathbf{I^{op}}}$ is also called the category of presheaves on I. We recall the construction of the category of elements of a presheaf P, denoted by $\int_{\mathbf{I}} P$. The objects of $\int_{\mathbf{I}} P$ are pairs (i,p), where i is an object of $\mathbf{I}$ and x is an element of $P(i)$. Its morphisms $(i',p') \to (i,p)$ are those morphisms $u : i' \to i$ of $\mathbf{I}$ for which $P(u) : P(i) \to P(i')$ satisfies $P(u)p = p'$. This category has a canonical projection functor $\pi_P : \int_{\mathbf{I}} P \to \mathbf{I}$ defined by $\pi_P(i,p) = i$.

The following result is proved in Theorem 2 of Chapter I of [**15**]:

THEOREM 4.2. *If $\chi : \mathbf{I} \to \mathbf{C}$ is a functor from a small category $\mathbf{I}$ to a cocomplete category $\mathbf{C}$, the functor R_χ from $\mathbf{C}$ to $\mathbf{Sets}^{\mathbf{I^{op}}}$ given by*

$$R_\chi(C) : i \rightsquigarrow Hom_{\mathbf{C}}(\chi(i), C)$$

has a left adjoint functor $L_\chi : \mathbf{Sets}^{\mathbf{I^{op}}} \to \mathbf{C}$ defined for each functor P in $\mathbf{Sets}^{\mathbf{I^{op}}}$ as the colimit:

$$L_\chi(P) = colim\left(\int_{\mathbf{I}} P \overset{\pi_P}{\to} \mathbf{I} \overset{\chi}{\to} \mathbf{C}\right)$$

Using this theorem, if we consider for each exterior space X, the functor $B(X) : \mathbf{\Delta} \to \mathbf{E}$ given by $B(X)([n]) = X \bar{\times} |\Delta[n]\,|$, then there is an associated left adjoint $L_{B(X)}$: $\mathbf{Sets}^{\mathbf{\Delta^{op}}} \longrightarrow \mathbf{E}$ which carries a simplicial set K to the exterior space

$$L_{B(X)}(K) = colim\left(\int_{\Delta} K \xrightarrow{\pi_K} \mathbf{I} \xrightarrow{B(X)} \mathbf{E}\right)$$

Therefore one has the adjunction isomorphism

$$Hom_{\mathbf{E}}\big(L_{B(X)}(K), Y\big) \cong Hom_{\mathbf{SS}}(K, \underline{Hom}_{\mathbf{E}}(X, Y))$$

where K is a finite simplicial set, and X, Y exterior spaces.

The following lemma will be useful in proving that $\mathbf{E}$ satisfies SM0:

LEMMA 4.2. *Let X be an exterior space and suppose that $F : \mathbf{J} \to \mathbf{T}$ is a functor such that $\mathbf{J}$ is a finite category, each $F(i)$ and $colim_{i\in\mathbf{J}}\, F(i)$ are compact Hausdorff spaces. Then*

$$colim_{i\in\mathbf{J}}(X \bar{\times} F(i)) \cong X \bar{\times} colim_{i\in\mathbf{J}}\, F(i).$$

PROOF: It is an immediate consequence of the properties of the adjunction isomorphisms given in Theorem 3.2. ∎

The following properties of colimits will be useful.

Given a functor $L : \mathbf{J}' \to \mathbf{J}$ and an object $j \in \mathbf{J}$, the comma category $j \downarrow L$ has as objects morphisms of the form $u : j \to L(j')$. A morphism from $u_0 : j \to L(j'_0)$ to $u_1 : j \to L(j'_1)$ is a morphism $v' : j'_0 \to j'_1$ which satisfies $L(v')u_0 = u_1$.

A category $\mathbf{J}$ is called connected if, given any two objects j_0 , j_1 in $\mathbf{J}$, there is a finite sequence of arrows (both directions possible) joining j_0 to j_1. A functor $L : \mathbf{J}' \to \mathbf{J}$ is final if for each j in $\mathbf{J}$, the comma category $j \downarrow L$ is nonempty and connected. For more details concerning final functors, we refer the reader to **[14]** and **[7]**. We shall use the following:

PROPOSITION 4.3. *If $L : \mathbf{J}' \to \mathbf{J}$ is final and $F : \mathbf{J} \to \mathbf{C}$ is a functor such that colim FL exists, then colim F exists and the canonical map colim $FL \to$ colim F is an isomorphism.*

Denote by Δ/n the full subcategory of Δ determined by the objects $[0], [1], \ldots, [n]$. Given a simplicial set K, one defines the functor $Sk_n(K)$ as the composite $(\mathbf{\Delta}/\mathbf{n})^{\mathbf{op}} \hookrightarrow \mathbf{\Delta}^{\mathbf{op}} \xrightarrow{K} \mathbf{Sets}$.

PROPOSITION 4.4. *If K is a simplicial set, with $dim(K) \leqslant n$, then the canonical functor $I : \int_{\mathbf{\Delta}/\mathbf{n}} Sk_n(K) \to \int_{\mathbf{\Delta}} K$ is final.*

Next, we shall see that $\mathbf{E}$, the category of exterior spaces, has a closed simplicial model category structure.

PROPOSITION 4.5. $\mathbf{E}$ *satisfies* SM0 *and* SM7.

PROOF: In order to prove SM0, if X is an exterior space and K is a finite simplicial set, we have the following:

$$\begin{aligned} X \otimes K &= X \bar{\times} \left| colim \left(\int_{\boldsymbol{\Delta}} K \overset{\pi_K}{\to} \boldsymbol{\Delta} \overset{y}{\to} \mathbf{Sets}^{\boldsymbol{\Delta}^{\mathrm{op}}} \right) \right| \\ &\cong X \bar{\times} colim \left(\int_{\boldsymbol{\Delta}} K \overset{y\pi_K}{\to} \mathbf{Sets}^{\boldsymbol{\Delta}^{\mathrm{op}}} \overset{|.|}{\to} \mathbf{T} \right) \\ &\cong X \bar{\times} colim \left(\int_{\boldsymbol{\Delta}/\mathbf{n}} Sk_n(K) \overset{I}{\to} \int_{\Delta} K \overset{|.|y\pi_K}{\longrightarrow} \mathbf{T} \right) \\ &\cong colim \left(\int_{\boldsymbol{\Delta}/\mathbf{n}} Sk_n(K) \overset{I}{\to} \int_{\Delta} K \overset{|.|y\pi_K}{\longrightarrow} \mathbf{T} \overset{X\bar{\times}-}{\to} \mathbf{E} \right) \\ &\cong colim \left(\int_{\boldsymbol{\Delta}} K \overset{B(X)\pi_K}{\to} \mathbf{E} \right) = L_{B(X)}(K). \end{aligned}$$

Observe that each simplicial set K is isomorphic to $colim \left(\int_{\boldsymbol{\Delta}} K \overset{y\pi_K}{\to} \mathbf{Sets}^{\boldsymbol{\Delta}^{\mathrm{op}}} \right)$. On the other hand, the realisation functor $|.| : \mathbf{Sets}^{\boldsymbol{\Delta}^{\mathrm{op}}} \to \mathbf{T}$ preserves colimits because it is a left adjoint. So we have a natural bijection on X, Y and K :

$$Hom_{\mathbf{E}}(X \otimes K, Y) \cong Hom_{\mathbf{SS}}\left(K, \underline{Hom}_{\mathbf{E}}(X, Y)\right)$$

Furthermore, $X \otimes (K \times L) = X \bar{\times} |K \times L| \cong X \bar{\times} (|K| \times |L|) = (X \otimes K) \otimes L$.
As a consequence we have:

$$\begin{aligned} Hom_{\mathbf{E}}\left(X, Y^{K \times L}\right) &\cong Hom_{\mathbf{E}}\left(X \otimes (L \times K), Y\right) \\ &\cong Hom_{\mathbf{E}}\left((X \otimes L) \otimes K, Y\right) \\ &\cong Hom_{\mathbf{E}}\left(X \otimes L, Y^K\right) \\ &\cong Hom_{\mathbf{E}}\left(X, \left(Y^K\right)^L\right) \end{aligned}$$

So we have that $Y^{K\times L} \cong \left(Y^K\right)^L$, $Hom_{\mathbf{E}}\left(Y, X^K\right) \cong Hom_{\mathbf{E}}(Y \otimes K, X) \cong Hom_{\mathbf{SS}}(K, \underline{Hom}_{\mathbf{E}}(Y, X))$ and SM0 is satisfied.

We now prove SM7. We shall see that $\mathbf{E}$ satisfies SM7(a). Let $p : X \to Y$ be an e-fibration (respectively, trivial e-fibration): Since $(.)^{\mathbf{N}}$ is a right adjoint, it follows that $(.)^{\mathbf{N}}$ preserves pull-backs. Moreover, we have that $\left(X^Y\right)^{\mathbf{N}} \cong \left(X^{\mathbf{N}}\right)^Y$ in $\mathbf{T}$, for every exterior space X and every locally compact space Y . From these remarks one has that the e-fibration (respectively, trivial e-fibration) p satisfies SM7(a) if and only if the fibration $p^{\mathbf{N}}$ of $\mathbf{T}$ satisfies SM7(a). However, the last statement follows from the fact that $\mathbf{T}$ has a closed simplicial model category structure. ∎

With SM0 and SM7 we complete the proof of our main Theorem 4.1.

5. Applications

A. Whitehead theorem for $\mathbb{N}$-complexes.

Consider again the exterior spaces $\mathfrak{S}^{n-1} = \mathbb{N}\bar{\times}S^{n-1}$, for $n = 0$ take $\mathfrak{S}^{-1} = \emptyset$, and $\mathfrak{D}^n = \mathbb{N}\bar{\times}D^n$ provided with the usual topologies and the externologies given by the corresponding families of complements of closed-compact subsets. Using sums and push-outs, we can construct the following spaces:

Definition 5.1: An $\mathbb{N}$-complex consists of an exterior space X with a filtration $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X$ such that X is the colimit of the filtration and for $n \geqslant 0$, X_n is obtained from X_{n-1} by a push-out of the form

$$\begin{array}{ccc} \coprod_{\alpha\in A} \mathfrak{S}^{n-1}_\alpha & \xrightarrow{\coprod_{\alpha\in A}\varphi_\alpha} & X_{n-1} \\ \downarrow & & \downarrow i_n \\ \coprod_{\alpha\in A} \mathfrak{D}^{n}_\alpha & \xrightarrow[\coprod_{\alpha\in A}\psi_\alpha]{} & X_n \end{array}$$

The subspace $\psi_\alpha(\mathfrak{D}^n_\alpha)$ will be called an n-dimensional $\mathbb{N}$-cell of X and $\varphi_\alpha : \mathfrak{S}^{n-1}_\alpha \to X_{n-1}$ will be called the attaching map of $\mathbb{N}$-cell $\psi_\alpha(\mathfrak{D}^n_\alpha)$.

We note that from the definition of $\mathbb{N}$-complex it follows that every $\mathbb{N}$-complex is a cofibrant exterior space. It is clear that for cofibrant exterior spaces right homotopies, left homotopies and exterior homotopies (Definition 3.3) induce the same relations between maps. From Quillen [**20**], one has that the homotopy category $\pi_0(\mathbf{E_{cof}})$ is equivalent to the localised category $\mathbf{Ho}(\mathbf{E})$, then one has

Theorem 5.1. *Let X, Y be $\mathbb{N}$-complexes and let $f : X \to Y$ be an exterior map. Then f is a homotopy exterior equivalence if and only if f is a weak exterior equivalence.*

B. Embedding and Whitehead theorems for proper homotopy categories.

The full embeddings $\mathbf{P} \hookrightarrow \mathbf{E}$, $\mathbf{P}^{\mathbb{N}} \hookrightarrow \mathbf{E}^{\mathbb{N}}$ induce full embeddings $\pi_0(\mathbf{P}) \hookrightarrow \pi_0(\mathbf{E})$, $\pi_0\left(\mathbf{P}^{\mathbb{N}}\right) \hookrightarrow \pi_0\left(\mathbf{E}^{\mathbb{N}}\right)$, where $\pi_0(\mathbf{C})$ denotes the category obtained by dividing morphisms by the corresponding homotopy relations.

Suppose that X is a locally finite CW-complex with finite dimension d and for each $0 \leqslant k \leqslant d$ either X has no k-cells or X has an infinite countable number of k-cells. Under these conditions X with the externology ε^X_{cc} admits the structure of a finite $\mathbb{N}$-complex. Taking into account that $\mathbb{N}$-complexes are cofibrant one has

Theorem 5.2. *Let X, Y be finite $\mathbb{N}$-complexes. Then*

$$Hom_{\pi_0(\mathbf{P})}(X,Y) \cong Hom_{\mathbf{Ho}(\mathbf{E})}(X_e, Y_e)$$

that is, $e : \mathbf{P} \hookrightarrow \mathbf{E}$ induces a full embedding $\pi_0(\mathbf{P}_{f\mathbb{N}}) \hookrightarrow \mathbf{Ho}(\mathbf{E})$, where $\pi_0(\mathbf{P}_{f\mathbb{N}})$ denotes the proper homotopy category of finite $\mathbb{N}$-complexes.

THEOREM 5.3. *Let $f : X \to Y$ be a proper map between finite $\mathbb{N}$-complexes. Then f is a proper homotopy equivalence if and only if $f_e = f$ is a weak exterior equivalence.*

REMARK. If X is an object in $\mathbf{P}^{\mathbf{N}}$, then the homotopy groups $\pi_k^e(X_e)$ are a global version of Brown's proper homotopy groups, see [**4**]. The differences are that we are using proper maps instead of germs of proper maps and we consider a base sequence instead of a base ray.

C. LOCALISED CATEGORY OF EXTERIOR SPACES AND STRONG SHAPE THEORY.

In some geometric contexts such as knot theory or Riemannian geometry, one frequently studies the classification of some families of pairs under isomorphisms (diffeomorphisms, isometries, et cetera).

The following category **SPairs** will be useful in these contexts. The objects in **SPairs** are pairs (X, A) of topological spaces and the morphisms are saturated maps, that is, maps $f : (X, A) \to (Y, B)$ such that $f^{-1}(B) \subset A$. In a natural way we have a notion of saturated homotopy and the associated homotopy category $\pi_0(\mathbf{SPairs})$.

Note that we have a functor $\mathbf{SPairs} \longrightarrow \mathbf{E}$ which carries a map $f : (X, A) \longrightarrow (Y, B)$ to $f \mid X \setminus A : X \setminus A \longrightarrow Y \setminus B$, where $X \setminus A$ has the relative topology and externology given by $\varepsilon_{X\setminus A} = \{U \setminus A \;/\; U \in \tau_X,\ A \subset U\}$ and similarly for $Y \setminus B$. This functor induces a functor $\pi_0(\mathbf{SPairs}) \to \pi_0(\mathbf{E})$ and a full embedding $\mathbf{Ho}(\mathbf{SPairs}) \to \mathbf{Ho}(\mathbf{E})$, where $\mathbf{Ho}(\mathbf{SPairs})$ is the category of pairs and saturated maps localised by maps $f : (X, A) \to (Y, B)$ such that $f \mid X \setminus A : X \setminus A \longrightarrow Y \setminus B$ is a weak exterior homotopy equivalence in $\mathbf{E}$. For example if K and L are knots in S^3 then we can consider the associated exterior spaces $S^3 \setminus K$ and $S^3 \setminus L$ and we can say that K and L are equivalent if $S^3 \setminus K$ and $S^3 \setminus L$ are isomorphic in $\mathbf{Ho}(\mathbf{E})$. In the case of smooth knots the study of this notion reduces to invariants and tools of standard homotopy theory, however these exterior spaces have more interest for the case of wild knots.

As a new application, we are going to show that the study of the strong shape category of Hausdorff compact metrisable spaces can be formulated in terms of exterior spaces and the corresponding localised category. For a formulation of the strong shape category we refer the reader to [**9**], other references on the strong shape theory are [**19, 17, 16**]. In particular we are going to use the approach to category **SSh** of the strong shape of compact metrisable spaces given by Edwards-Hastings [**9**], where **SSh** is introduced by using the telescopic construction.

We know that any compact metric A with finite covering dimension can be considered as a subspace of the interior $Int(I^{n-1})$ of some $(n-1)$-cube. Suppose that $A \subset Int(I^{n-1}) \subset I^{n-1} \subset I^n$ and $B \subset Int(I^{m-1}) \subset I^{n-1} \subset I^m$, where $I^k \hookrightarrow I^{k+1}$ denotes the inclusion $(t_1, \cdots, t_k) \rightsquigarrow (0, t_1, \cdots, t_k)$. Note that pairs (I^n, A) (I^m, B) are carried by the functor $\mathbf{SPairs} \to \mathbf{E}$ to exterior spaces $I^n \setminus A$ and $I^m \setminus B$. With this notation, one can prove the following:

THEOREM 5.4. *Let A, B compact spaces and suppose that we have inclusions*

$$A \subset Int\left(I^{n-1}\right) \subset I^{n-1} \subset I^n, \; B \subset Int\left(I^{m-1}\right) \subset I^{n-1} \subset I^m.$$

Then we have:

(i) There exists a bijection $Hom_{\mathbf{SSh}}(A,B) \cong Hom_{\mathbf{Ho}(\mathbf{E})}(I^n \setminus A, I^m \setminus B)$.

(ii) A, B have the same strong shape type if and only if $I^n \setminus A$ is isomorphic to $I^m \setminus B$ in $\mathbf{Ho}(\mathbf{E})$.

PROOF: Take neighbourhoods of A in I^{n-1} and of B in I^{m-1}

$$N_0^A \supset N_1^A \supset N_2^A \supset \cdots \supset A$$
$$N_0^B \supset N_1^B \supset N_2^B \supset \cdots \supset B.$$

Denote by $Q = \prod_{\mathbb{N}} I$ the Hilbert cube and take the inclusions $I^k \hookrightarrow Q$, $(t_1, \cdots, t_k) \rightsquigarrow (t_1, \cdots, t_k, 1/2, 1/2, \cdots)$, for $k = n-1$ or $k = m-1$. Using these inclusions we can suppose that A and B are subspaces of the pseudointerior of the Hilbert cube. We can choose neighbourhoods of A, B in Q

$$U_0^A \supset U_1^A \supset U_2^A \supset \cdots \supset A$$
$$U_0^B \supset U_1^B \supset U_2^B \supset \cdots \supset B$$

such that $U_i^A = N_i^A \times M_i^A$, $U_i^B = N_i^B \times M_i^B$ where M_i^A , M_i^B are contractible compact spaces. The projections $U_i^A \to N_i^A$, $U_i^B \to N_i^B$ are homotopy equivalences that induce proper homotopy equivalences on the telescopic constructions $Tel\left(\{U_i^A\}\right) \to Tel\left(\{N_i^A\}\right)$, $Tel\left(\{U_i^B\}\right) \to Tel\left(\{N_i^B\}\right)$. Then one has

$$\begin{aligned}
&(1) \qquad & Hom_{\mathbf{SSh}}(A,B) &= Hom_{\pi_0(\mathbf{P})}\left(Tel\left(\{U_i^A\}\right), Tel\left(\{U_i^B\}\right)\right) \\
&(2) & &\cong Hom_{\pi_0(\mathbf{P})}\left(Tel\left(\{N_i^A\}\right), Tel\left(\{N_i^B\}\right)\right) \\
&(3) & &\cong Hom_{\pi_0(\mathbf{E})}(I^n \setminus A, I^m \setminus B) \\
&(4) & &\cong Hom_{\mathbf{Ho}(\mathbf{E})}(I^n \setminus A, I^m \setminus B)
\end{aligned}$$

where for (1) we consider the definition of the hom-set in the strong shape category using telescopes, see [**9**, page 231], (2) follows from the proper homotopy equivalences between the corresponding telescopes, to obtain isomorphism (3) we consider $Tel\left(\{N_i^A\}\right)$, $Tel\left(\{N_i^B\}\right)$ as subspaces of I^n and I^m, respectively, isomorphism (4) comes from the fact that $I^n \setminus A$, $I^m \setminus B$ admit the structure of a finite $\mathbb{N}$-complex, and therefore we can apply Theorem 5.2. ∎

D. Shape, strong shape and discrete (inward) shape theories and exterior homotopy categories.

In order to give the set of strong shape morphisms as a hom-set in the category $\mathbf{Ho(E)}$, the hom-set of the strong shape category of compact metric spaces has been defined by using a telescopic construction in subsection above. Nevertheless, we also want to consider the notion of "approaching map" given by Quigley [**19**] and the corresponding approaching homotopy theory. Both formulations of the strong shape category of compact metric spaces are equivalent. We refer the reader to [**6**] for a proof of this fact. In this paper, Cathey considered several representations of the strong shape category. In particular, [**6**, Theorem 2.14] gives a representation using telescopic constructions and [**6**, Theorem 2.13 and Lemma 2.12] prove that the hom-set can be also represented by approaching maps. In the subsection above the hom-set based in the telescopic construction has been denoted by $Hom_{\mathbf{SSh}}(A,B)$. In this subsection, to recall that we are thinking on a different formulation of the strong shape category we shall denote the hom-set by $\mathbf{Sh}(A,B)$. Of course using the equivalence of categories between both formulations, for compact metric spaces one has natural isomorphisms $Hom_{\mathbf{SSh}}(A,B) \cong \mathbf{Sh}(A,B)$.

It is well known that a compact metrisable Y is homeomorphic to a closed subspace of the Hilbert cube Q. Given a continuous map $i : Y \to Q$ such that $i : Y \to i(Y)$ is a homeomorphism, we can consider the exterior space $Q^i_Y = Q$ whose externology is given by those open subsets U of Q such that $i(Y) \subset U \subset Q$. Given two continuous maps $i : Y \to Q$, $j : Y \to Q$ such that $i : Y \to i(Y)$, $j : Y \to j(Y)$ are homeomorphisms, one can use that Q is an absolute retract and locally convex to check that Q^i_Y, Q^j_Y have the same exterior homotopy type. Because the exterior homotopy type does not depend on the given "inclusion", the exterior space above will be denoted by Q_Y.

If we consider the functor $\mathbf{E}^{op} \times \mathbf{E} \to \mathbf{T}$, $(X,Z) \rightsquigarrow Z^X$, given in section 3, one obtains the spaces $Q^{\mathbb{N}}_Y$ and $Q^{\mathbb{R}_+}_Y$, where $\mathbb{R}_+$ is the subspace of non negative real numbers provided with the externology $\varepsilon^{\mathbb{R}_+}_{cc}$. In the case that we have chosen a base point $y_0 \in Y \subset Q$, we can consider the exterior maps

$$\begin{array}{ll} \mathbb{N} \to Q_Y, & n \rightsquigarrow y_0, n \in \mathbb{N} \\ \mathbb{R}_+ \to Q_Y, & r \rightsquigarrow y_0, r \in \mathbb{R}_+ \end{array}$$

as base points of the spaces $Q^{\mathbb{N}}_Y$ and $Q^{\mathbb{R}_+}_Y$, respectively.

We also consider the inclusion map $in : \mathbb{N} \to \mathbb{R}_+$ and the induced maps

$$\begin{array}{rcl} in^* & : & Q^{\mathbb{R}_+}_Y \to Q^{\mathbb{N}}_Y \\ in \bar{\times} id_X & : & \mathbb{N} \bar{\times} X \to \mathbb{R}_+ \bar{\times} X \end{array}$$

where X is a compact metrisable space and $\mathbb{N} \bar{\times} X$, $\mathbb{R}_+ \bar{\times} X$ have the externologies given in Section 3.

Given X, Y compact metrisable spaces, we can use exterior maps $\mathbb{R}_+ \bar{\times} X \to Q_Y$ instead of Quigley's approaching maps of the form $\mathbb{R}_+ \times Q_X \to Q_Y$ to represent strong shape morphisms. Therefore the set of strong shape morphism from X to Y can be defined by

$$\mathbf{SSh}(X,Y) = \pi_0\mathbf{E}(\mathbb{R}_+\bar{\times}X, Q_Y) \cong \pi_0\mathbf{T}\left(X, Q_Y^{\mathbb{R}_+}\right)$$

and the set of discrete (inward) shape morphisms by

$$\mathbf{DSh}(X,Y) = \pi_0\mathbf{E}(\mathbb{N}\bar{\times}X, Q_Y) \cong \pi_0\mathbf{T}\left(X, Q_Y^{\mathbb{N}}\right)$$

where we have applied the exponential law given in Theorem 3.2(i).

The set of shape morphisms from X to Y is given by

$$\mathbf{Sh}(X,Y) = Image\left(\pi_0\mathbf{E}(\mathbb{R}_+\bar{\times}X, Q_Y) \stackrel{(in\bar{\times}id_X)^*}{\longrightarrow} \pi_0\mathbf{E}(\mathbb{N}\bar{\times}X, Q_Y)\right)$$

or equivalently

$$\mathbf{Sh}(X,Y) \cong Image\left(\pi_0\mathbf{T}\left(X, Q_Y^{\mathbb{R}_+}\right) \stackrel{(in^*)_*}{\longrightarrow} \pi_0\mathbf{T}\left(X, Q_Y^{\mathbb{N}}\right)\right)$$

If we consider the shift operator $s_X : \mathbb{N}\bar{\times}X \to \mathbb{N}\bar{\times}X$, $s_X(n,x) = (n+1,x)$, $(n,x) \in \mathbb{N}\bar{\times}X$, then an alternative definition of $\mathbf{Sh}(X,Y)$ can be given as the equaliser

$$\mathbf{Sh}(X,Y) \longrightarrow \pi_0\mathbf{E}(\mathbb{N}\bar{\times}X, Q_Y) \underset{id}{\overset{s_X^*}{\rightrightarrows}} \pi_0\mathbf{E}(\mathbb{N}\bar{\times}X, Q_Y)$$

For the pointed setting, given $x_0 \in X \subset Q$ and $y_0 \in Y \subset Q$, we consider the category $\mathbf{E}^{\mathbb{R}_+}$ and the objects $\mathbb{R}_+\bar{\times}X$, Q_Y with the base rays

$$\begin{aligned} \mathbb{R}_+ \to \mathbb{R}_+\bar{\times}X\ , &\quad r \rightsquigarrow (r, x_0), r \in \mathbb{R}_+\ , \\ \mathbb{R}_+ \to Q_Y\ , &\quad r \rightsquigarrow y_0, r \in \mathbb{R}_+\ , \end{aligned}$$

then the set of pointed strong shape morphisms is given by

$$\mathbf{SSh}^*(X,Y) = \pi_0\mathbf{E}^{\mathbb{R}_+}(\mathbb{R}_+\bar{\times}X, Q_Y) \cong \pi_0\mathbf{T}^*\left(X, Q_Y^{\mathbb{R}_+}\right).$$

We take the category $\mathbf{E}^{\mathbb{N}}$ and the objects $\mathbb{N}\bar{\times}X$, Q_Y with the sequences

$$\begin{aligned} \mathbb{N} \to \mathbb{N}\bar{\times}X &\quad n \rightsquigarrow (n, x_0),\ n \in \mathbb{N} \\ \mathbb{N} \to Q_Y &\quad n \rightsquigarrow y_0,\ n \in \mathbb{N}, \end{aligned}$$

to define the set of pointed discrete shape morphisms as

$$\mathbf{DSh}^*(X,Y) = \pi_0\mathbf{E}^{\mathbb{N}}(\mathbb{N}\bar{\times}X, Q_Y) \cong \pi_0\mathbf{T}^*\left(X, Q^{\mathbb{N}_Y}\right).$$

The inclusion $in : \mathbb{N} \to \mathbb{R}_+$ induces a canonical functor $\pi_0\mathbf{E}^{\mathbb{R}_+} \to \pi_0\mathbf{E}^{\mathbb{N}}$ that can be used to define the set of pointed shape morphisms as

$$\begin{aligned} \mathbf{Sh}^*(X,Y) &= Image\left(\pi_0\mathbf{E}^{\mathbb{R}_+}(\mathbb{R}_+\bar{\times}X, Q_Y) \to \pi_0\mathbf{E}^{\mathbb{N}}(\mathbb{N}\bar{\times}X, Q_Y)\right) \\ &\cong Image\left(\pi_0\mathbf{T}^*\left(X, Q_Y^{\mathbb{R}_+}\right) \to \pi_0\mathbf{T}^*\left(X, Q_Y^{\mathbb{N}}\right)\right) \end{aligned}$$

In order to compare the different sets of shape morphisms, we can consider the (Serre) fibration $in^* : Q_Y^{\mathbb{R}_+} \to Q_Y^{\mathbb{N}}$. First we note that the homotopy fibre of this map is homeomorphic to the space $\Omega\big(Q_Y^{\mathbb{N}}\big)$ which is the mapping space of Hawai earrings close to Y in Q and based at $y_0 \in Y$. Therefore one has the fibre sequence

$$\ldots \to \Omega\big(Q_Y^{\mathbb{N}}\big) \overset{S-I}{\to} \Omega\big(Q_Y^{\mathbb{N}}\big) \to Q_Y^{\mathbb{R}_+} \to Q_Y^{\mathbb{N}}$$

where the map $\Omega\big(Q_Y^{\mathbb{N}}\big) \overset{S-I}{\to} \Omega\big(Q_Y^{\mathbb{N}}\big)$ is given as follows: An element of $\Omega\big(Q_Y^{\mathbb{N}}\big)$ is determined by a pointed exterior map $\alpha : \mathbb{N}\bar{\times}S^1 \to Q_Y$, let $\alpha_n : S^1 \to Q_Y$ denote the map $\alpha_n(x) = \alpha(n,x)$. The operator $S - I$ maps the element represented by α into an element β such that $\beta_n = \alpha_{n+1} \cdot \alpha_n^{-1}$, where $\alpha_{n+1} \cdot \alpha_n^{-1}$ denotes the usual product path and α_n^{-1} is the inverse path.

Applying the functor $\pi_0\mathbf{T}^*(X,-)$ to the sequence above, one has the following:

Theorem 5.5. *Let X, Y be compact metrisable spaces, then the following sequence is exact:*

$\ldots \to \mathbf{DSh}^*(\Sigma^p X,Y) \overset{S-I}{\to} \mathbf{DSh}^*(\Sigma^p X,Y) \to \mathbf{SSh}^*(\Sigma^{p-1}X,Y) \to \mathbf{DSh}^*(\Sigma^{p-1}X,Y) \to$
$\ldots \to \mathbf{DSh}^*(\Sigma X,Y) \overset{S-I}{\to} \mathbf{DSh}^*(\Sigma X,Y) \to \mathbf{SSh}^*(X,Y) \to \mathbf{DSh}^*(X,Y)$
where for $p \geqslant 0$ one has that

$$\begin{aligned} \mathbf{Sh}^*(\Sigma^p X,Y) &\cong Image\big(\mathbf{SSh}^*(\Sigma^p X,Y) \to \mathbf{DSh}^*(\Sigma^p X,Y)\big) \\ &\cong Ker\big(\mathbf{DSh}^*(\Sigma^p X,Y) \to \mathbf{DSh}^*(\Sigma^p X,Y)\big)\,. \end{aligned}$$

Remark. (1) If in the sequence above, we take $X = S^0$, then $\mathbf{Sh}^*(S^n,Y)$ is the n-th fundamental group defined by Borsuk. The group $\mathbf{SSh}^*(S^n,Y)$ is Quigley's approaching group and $\mathbf{DSh}^*(S^n,Y)$ corresponds to Quigley's inward group. We note that from the exact sequence above one obtains easily Quigley's exact sequence **[19]**.

(2) From the last terms of the exact sequence of the theorem, one obtains the short exact sequence

$$0 \to Coker(S-I) \to \mathbf{SSh}^*(X,Y) \to \mathbf{Sh}^*(X,Y) \to 0$$

which is a version of the Comparison Theorem of Edwards-Hastings [**9**], obtained without using the Bousfield-Kan spectral sequence of a tower of fibrations.

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