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Sequential homology ☆

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Abstract

The notion of exterior space consists of a topological space together with a certain nonempty family of open subsets that is thought of as a 'system of open neighborhoods at infinity'. An exterior map is a continuous map which is 'continuous at infinity'. A strongly locally finite CW-complex X, whose skeletons are provided with the family of the complements of compact subsets, can be considered as an exterior space $\overline{X}$. Associated with a compact metric space we also consider the open fundamental complex $\overline{OFC(X)}$ introduced by Lefschetz.

In this paper we use sequences of cycles converging to infinity to introduce 'ordinary' sequential homology and cohomology theories in the category of exterior spaces. One of the interesting differences with respect to the ordinary theories of topological spaces is that the role of a point is played by the exterior space $\mathbb{N}$ of natural numbers with the discrete topology and the cofinite externology.

For a strongly locally finite CW-complex X, we see that the singular homology of X is isomorphic to $H^{seq}_\bullet(\overline{X}; \bigoplus_0^\infty \mathbb{Z})$, the locally finite homology is isomorphic to $H^{seq}_\bullet(\overline{X}; \prod_0^\infty \mathbb{Z})$ and the end homology is isomorphic to $H^{seq}_\bullet(\overline{X}; \prod_0^\infty \mathbb{Z}/\bigoplus_0^\infty \mathbb{Z})$. For cohomology one has that the compact support cohomology is isomorphic to $H_{seq}^\bullet(\overline{X}; \bigoplus_0^\infty \mathbb{Z})$, the singular cohomology is isomorphic to $H_{seq}^\bullet(\overline{X}; \prod_0^\infty \mathbb{Z})$ and the end cohomology is isomorphic to $H_{seq}^\bullet(\overline{X}; \prod_0^\infty \mathbb{Z}/\bigoplus_0^\infty \mathbb{Z})$.

With respect to the Lefschetz fundamental complex, one has that the Čech homology of a compact metric space can be found as a subgroup of $H^{seq}_\bullet(\overline{OFC(X)}; \mathcal{R})$, the Steenrod homology is isomorphic to $H^{seq}_{\bullet+1}(\overline{OFC(X)}; \prod_0^\infty \mathbb{Z}/\bigoplus_0^\infty \mathbb{Z})$ and the Čech cohomology of X is isomorphic to $H_{seq}^\bullet(\overline{OFC(X)}; \prod_0^\infty \mathbb{Z}/\bigoplus_0^\infty \mathbb{Z})$.

Finally, one also has a Poincaré isomorphism $H^q_{seq}(\overline{M}) \cong H^{seq}_{n-q}(\overline{M})$, where M is a triangulable, second countable, orientable, n-manifold. We remark that in both sides of the isomorphism we are using sequential theories.

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Introduction

The proper category of spaces and proper maps is a suitable framework for the study of noncompact spaces. Nevertheless, one of the problems of this category is that it does not have enough limits and colimits to develop the usual homotopy constructions such as homotopy fibres and loop spaces. Recently, the authors together with García Pinillos [8] have given a solution using the notion of exterior space. The category $\boldsymbol{E}$ of exterior spaces is complete and cocomplete and contains the proper category as a full subcategory.

An *exterior space* $(X, \varepsilon \subset \tau)$, consists of a topological space (X, τ) together with a nonempty family of open subsets ε, called *externology*, which is closed by finite intersections and such that if U is an open subset and $U \supset E$, $E \in \varepsilon$, then $U \in \varepsilon$.

In this paper we consider analogues of the ordinary homology and cohomology theories defined for the category of pairs of exterior spaces.

An *ordinary homology theory* from the category of pairs $\boldsymbol{E}^{(\mathbf{2})}$ to an Abelian category $\mathcal{A}$ is a pair (H, ∂), where H consists of a family of functors

$$H_q : \boldsymbol{E}^{(\mathbf{2})} \to \mathcal{A}, \quad q \in \mathbb{Z},$$

and ∂ is a family of natural transformations

$$\partial = \partial_q : H_q(X, A) \to H_{q-1}(A), \quad q \in \mathbb{Z},$$

satisfying certain basic properties analogous to the Eilenberg–MacLane axioms of ordinary homology. A *cohomology theory* from $\boldsymbol{E}^{(\mathbf{2})}$ to an Abelian category $\mathcal{A}$ is just a homology theory from $\boldsymbol{E}^{(\mathbf{2})}$ to the opposite category $\mathcal{A}^{op}$.

These theories are called ordinary because for certain cellular spaces, called bi-complexes, the homology groups are determined by the coefficient group $H_0^{seq}(\mathbb{N})$, where $\mathbb{N}$ is the exterior space of nonnegative integers. A bi-complex consists of an exterior space X together with a filtration $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X$, such that X is the colimit of the skeletons X_n of the filtration. The n-skeleton X_n is obtained from the $(n-1)$-skeleton X_{n-1} by attaching single n-cells, D^n, where the externology agrees with the usual topology and noncompact n-cells, $D^n \bar{\times} \mathbb{N}$, that have the usual topology and the cocompact externology. We note that a strongly locally finite CW-complex X, whose skeletons have the externology of the complements of compact subsets, has the structure of a bi-complex $\overline{X}$ having, for each $n \geqslant 0$, its n-skeleton consisting of a finite number of cells. We remark that X_n, as a CW-complex, can have an infinite number of standard cells. The importance of bi-complexes having finite n-skeletons is that the homology is isomorphic to the corresponding cellular homology which is determined by the coefficient group.

In this context, we think that the most important ordinary homology theory for exterior spaces (which are first countable at infinity) is the theory introduced in this paper that we have called *sequential homology*. It is called sequential because its definition is based on sequences $c = (c_0, c_1, c_2, \ldots)$ of singular n-chains converging to infinity. An n-cycle is a sequence of singular n-cycles and an n-boundary a sequence of singular n-boundaries, in both cases converging to infinity. It is very interesting to remark that the Abelian group of n-chains admits the structure of an $\mathcal{R}$-module, where $\mathcal{R}$ is the ring of locally finite matrices (see [6]), whose elements are matrices with integral entries a_{ij}, where i, j are nonnegative integers, such that each file and column have a finite number of nonzero entries.

We have also introduced the homologies that we have called *tubular homology* and *closed tubular homology*. In the first case, the n-cycles are determined by a sequence of $(n-1)$-cycles $(z_0, z_1, z_2, \ldots)$ and a sequence of n-chains $c = (c_0, c_1, c_2, \ldots)$ in such a way that $\partial c_0 = z_0 - z_1,\ \partial c_1 = z_1 - z_2, \ldots$; that is, we have an 'infinite tube' with boundary z_0. Taking n-cycles with $z_0 = 0$, we have 'infinite closed tubes' that give rise to closed tubular homology.

In order to have sequential homology and cohomology with coefficients, it is convenient to have chain complexes that are projective at each dimension. We solve this problem using the structure of the closed model category of the category of chain complexes of $\mathcal{R}$-modules which are bounded below. This permits the use of cofibrant approximations to define homology and (cohomology) with coefficients in a left (right) $\mathcal{R}$-module. There are three $\mathcal{R}$-modules that play an important role in this theory: $\bigoplus_0^\infty \mathbb{Z}$, $\prod_0^\infty \mathbb{Z}$ and $\prod_0^\infty \mathbb{Z}/\bigoplus_0^\infty \mathbb{Z}$. The action of the ring on the left and on the right is given by matrix multiplication.

We have noted that for a bi-complex with finite n-skeletons, the sequential homology with coefficients in $\bigoplus_0^\infty \mathbb{Z}$ is the singular homology; taking coefficients in $\prod_0^\infty \mathbb{Z}$, one has the closed tubular homology; and using $\prod_0^\infty \mathbb{Z}/\bigoplus_0^\infty \mathbb{Z}$, one has the tubular homology. In this paper we also compare the new homologies and cohomologies for exterior spaces with the standard homologies and cohomologies.

Let X be a locally finite CW-complex and consider the associated bi-complex $\overline{X}$ where the n-skeleton is provided with the cocompact externology and in $\overline{X}$ we take the colimit externology $\overline{X} = \operatorname{colim} \overline{X}_n$.

We note the following relations for a strongly locally finite CW-complex X:

(i) the singular homology of X is isomorphic to the sequential homology of $\overline{X}$ with coefficients in $\bigoplus_0^\infty \mathbb{Z}$,

(ii) the locally finite homology of X is isomorphic to the closed tubular homology of $\overline{X}$ and to the sequential homology with coefficients in $\prod_0^\infty \mathbb{Z}$,

(iii) the end homology of X is isomorphic to the tubular homology of $\overline{X}$ and to the sequential homology with coefficients in $\prod_0^\infty \mathbb{Z}/\bigoplus_0^\infty \mathbb{Z}$,

(iv) the compact support cohomology of X is isomorphic to the sequential cohomology of $\overline{X}$ with coefficients in $\bigoplus_0^\infty \mathbb{Z}$,

(v) the singular cohomology of X is isomorphic to the sequential cohomology of $\overline{X}$ with coefficients in $\prod_0^\infty \mathbb{Z}$,

(vi) the end cohomology of X is isomorphic to the sequential cohomology of $\overline{X}$ with coefficients in $\prod_0^\infty \mathbb{Z}/\bigoplus_0^\infty \mathbb{Z}$.

Associated with a compact-metric space Y one also has the open fundamental complex $OFC(Y)$, introduced by Lefschetz [10] and also called the telescopic construction of Milnor [12], and the corresponding exterior space $\overline{OFC(Y)}$. We have then that

(vii) the Čech homology group of Y can be found as the subgroup of the $\mathcal{R}$-module of sequential homology of $\overline{OFC(Y)}$ annihilated by the element *id*-*Sh* of the ring $\mathcal{R}$ of locally finite matrices,

(viii) the nth Steenrod homology group of Y is isomorphic to the $(n+1)$th tubular homology group of $\overline{OFC(Y)}$ and to the $(n+1)$th sequential homology group of $\overline{OFC(Y)}$ with coefficients in $\prod_0^\infty \mathbb{Z}/\bigoplus_0^\infty \mathbb{Z}$, and the nth reduced Steenrod homology group of of Y is isomorphic to the $(n+1)$th closed tubular homology group of $\overline{OFC(Y)}$ and to the $(n+1)$th sequential homology group of $\overline{OFC(Y)}$ with coefficients in $\prod_0^\infty \mathbb{Z}$,

(ix) the Čech cohomology of Y is isomorphic to the sequential cohomology group of $\overline{OFC(Y)}$ with coefficients in $\prod_0^\infty \mathbb{Z}/\bigoplus_0^\infty \mathbb{Z}$, and the nth reduced Čech cohomology of Y is isomorphic to the $(n+1)$th sequential homology group of $\overline{OFC(Y)}$ with coefficients in $\bigoplus_0^\infty \mathbb{Z}$.

Finally, we observe that for a triangulable, second countable, orientable n-manifold M, the qth sequential homology of $\overline{M}$ is isomorphic to the $(n-q)$th sequential cohomology of $\overline{M}$. We remark that we use only sequential homology and sequential cohomology.

1. Preliminaries

1.1. The category of exterior spaces

Let X and Y be topological spaces. A continuous map $f : X \to Y$ is said to be *proper* if for every closed compact subset K of Y, $f^{-1}(K)$ is a compact subset of X. The category of spaces and proper maps will be denoted by $\boldsymbol{P}$. This category and the corresponding proper homotopy category are very useful for the study of noncompact spaces. Nevertheless, one has the problem that this category does not have enough limits and colimits, and so we cannot develop the usual homotopy constructions such as loops, homotopy limits and colimits, etc.

In [8] we gave a solution to this problem introducing the notion of exterior space. The category of exterior spaces and maps is complete and cocomplete, and contains as a full subcategory the category of spaces and proper maps. Furthermore, it has a closed simplicial model category structure in the sense of Quillen [15], hence it establishes a nice framework for the study of proper homotopy theory. We begin by recalling the notion of exterior space.

Roughly speaking, an exterior space is a topological space X with a neighbourhood system at infinity.

Definition 1.1. An *exterior space* (or exterior topological space) $(X, \varepsilon \subset \tau)$ consists of a space (X, τ) together with a nonempty collection ε of open subsets, called *externology*, satisfying:

(E1) if $E_1, E_2 \in \varepsilon$ then $E_1 \cap E_2 \in \varepsilon$,
(E2) if $E \in \varepsilon$, $U \in \tau$ and $E \subset U$ then $U \in \varepsilon$.

An open set E which is in ε is said to be an *exterior-open* subset or in short, an *e-open* subset. A map $f:(X, \varepsilon \subset \tau) \to (X', \varepsilon' \subset \tau')$ is said to be *exterior* if it is continuous and $f^{-1}(E) \in \varepsilon$, for all $E \in \varepsilon'$.

The category of exterior spaces and maps will be denoted by $\boldsymbol{E}$.

Given an space (X, τ), one can always consider the *trivial* exterior space by taking $\varepsilon = \{X\}$, and the *total* exterior space if one takes $\varepsilon = \tau$. In this paper, an important role will be played by the family ε_{cc}^X of the complements of closed-compact subsets of X.

There is a full embedding $e:\boldsymbol{P} \hookrightarrow \boldsymbol{E}$. It carries a space X to the exterior space X_e which is provided with the topology of X and ε_{cc}^X. A proper map $f:X \to Y$ is carried to the exterior map $f_e:X_e \to Y_e$ given by $f_e = f$.

Definition 1.2. Let $(X, \varepsilon \subset \tau)$ be an exterior space. An *exterior base* on $(X, \varepsilon \subset \tau)$ is a collection of e-open subsets, $\beta \subset \varepsilon$, such that for every e-open subset E there exists $B \in \beta$ such that $B \subset E$. If an exterior space X has a countable exterior base $\beta = \{E_n\}_{n=0}^{\infty}$ then we say that X is *first countable at infinity*.

Note that for these exterior spaces we can suppose, without loss of generality, that

$$X = E_0 \supset E_1 \supset E_2 \supset \cdots \supset E_n \supset \cdots .$$

Notice that every σ-compact space provided with ε_{cc}^X is first countable at infinity. An exterior base is very useful for checking when a continuous map $f:(X, \varepsilon \subset \tau) \to (X', \varepsilon' \subset \tau')$ between exterior spaces is exterior. It is sufficient to see that $f^{-1}(B) \in \varepsilon$ for all $B \in \beta'$, where β' is a given exterior base.

Definition 1.3. Let X be an exterior space and Y be a topological space. We consider on $X \times Y$ the product topology and the distinguished open subsets E of $X \times Y$, such that for each $y \in Y$ there exists $U_y \in \tau_Y$, $y \in U_y$ and $E_y \in \varepsilon_X$ such that $E_y \times U_y \subset E$. This exterior space will be denoted by $X \bar{\times} Y$.

This construction gives a functor $\boldsymbol{E} \times \mathbf{Top} \to \boldsymbol{E}$, where **Top** denotes the category of topological spaces. When Y is a compact space, we can prove that E is an e-open subset if and only if it is an open subset and there exists $G \in \varepsilon_X$ such that $G \times Y \subset E$. Furthermore, if Y is a compact space and $\varepsilon_X = \varepsilon_{cc}^X$, then $\varepsilon_{X \bar{\times} Y}$ coincides with the complements of all closed-compact subsets of $X \times Y$.

We will consider the set $\mathbb{N}$ of nonnegative integers with the discrete topology and the cofinite externology.

Let S^{n-1}, D^n be the $(n-1)$-sphere and the n-disc, respectively. We will let

$$\mathfrak{S}^{n-1} = \mathbb{N} \bar{\times} S^{n-1}, \quad \text{for } n \geqslant 1$$

and let $\mathfrak{S}^{-1} = \emptyset$. Similarly $\mathfrak{D}^n = \mathbb{N} \bar{\times} D^n$, $n \geqslant 0$.

Recall that a CW-*complex* is a space X with a filtration

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X$$

such that X is the colimit of the filtration and for $n \geqslant 0$, X_n is obtained from X_{n-1} by a push-out in **Top** of the form

$$\begin{array}{ccc}
\coprod_{\alpha\in A_n} S^{n-1}_\alpha & \xrightarrow{\coprod_{\alpha\in A_n}\varphi_\alpha} & X_{n-1} \\
\big\downarrow & & \big\downarrow{\scriptstyle i_n} \\
\coprod_{\alpha\in A_n} D^n_\alpha & \xrightarrow[\coprod_{\alpha\in A_n}\psi_\alpha]{} & X_n
\end{array}$$

The notion of $\mathbb{N}$-complex was introduced in [8]. It can be constructed from a discrete space provided with the complements of its finite subsets by consecutively attaching noncompact cells.

Definition 1.4. An $\mathbb{N}$-*complex* consists of an exterior space X with a filtration

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X$$

such that X is the colimit of the filtration and for $n \geqslant 0$, X_n is obtained from X_{n-1} by a push-out in $\boldsymbol{E}$ of the form

$$\begin{array}{ccc}
\coprod_{\alpha\in A_n} \mathfrak{S}^{n-1}_\alpha & \xrightarrow{\coprod_{\alpha\in A_n}\varphi_\alpha} & X_{n-1} \\
\big\downarrow & & \big\downarrow{\scriptstyle i_n} \\
\coprod_{\alpha\in A_n} \mathfrak{D}^n_\alpha & \xrightarrow[\coprod_{\alpha\in A_n}\psi_\alpha]{} & X_n
\end{array}$$

One can check that a locally finite CW-complex X with finite dimension d and, for each $0 \leqslant k \leqslant d$, either having no k-cells or having a countably infinite number of k-cells, provided with ε^X_{cc} admits the structure of a finite $\mathbb{N}$-complex. If X_n is obtained from X_{n-1} by a push-out in $\boldsymbol{E}$ of the form

$$\begin{array}{ccc}
(\coprod_{\alpha\in A_n} \mathfrak{S}^{n-1}_\alpha)\sqcup(\coprod_{\beta\in B_n} S^{n-1}_\beta) & \xrightarrow{(\coprod_{\alpha\in A_n}\varphi_\alpha)\sqcup(\coprod_{\beta\in B_n}\varphi_\beta)} & X_{n-1} \\
\big\downarrow & & \big\downarrow{\scriptstyle i_n} \\
(\coprod_{\alpha\in A_n} \mathfrak{D}^n_\alpha)\sqcup(\coprod_{\beta\in B_n} D^n_\beta) & \xrightarrow[(\coprod_{\alpha\in A_n}\psi_\alpha)\sqcup(\coprod_{\beta\in B_n}\psi_\beta)]{} & X_n
\end{array}$$

where in S^{n-1}_β, D^n_β we consider their topologies as externologies, we obtain the notion of *bi-complex*. It is not difficult to see that a countable locally finite, finite-dimensional CW-complex with externology ε^X_{cc} admits the structure of a finite bi-complex.

Definition 1.5. The subsets $\psi_\alpha(\mathfrak{D}^n_\alpha)$, $\psi_\beta(D^n_\beta)$ are called the *n-dimensional $\mathbb{N}$-cell* and *n-dimensional simple cell*, respectively. φ_α and φ_β are called the *attaching maps*.

1.2. Homology and cohomology of CW-complexes

In this paper we deal with several homologies and cohomologies for suitable CW-complexes as well as their corresponding cellular homologies and cohomologies.

1.2.1. Singular homology and cohomology

If X is a space, let $S_\bullet^{sin}(X)$ denote the singular chain complex and $H_n^{sin}(X) = H_n(S_\bullet(X))$ the singular homology of X. The cellular chain complex of a CW-complex X, $C_\bullet^{sin}(X)$, is given by

$$C_n^{sin}(X) = H_n^{sin}(X_n, X_{n-1}) \cong \bigoplus_{A_n} \mathbb{Z},$$

where A_n is the set of n-cells of X and the boundary homomorphism is the composite

$$H_n^{sin}(X_n, X_{n-1}) \xrightarrow{\partial} H_{n-1}^{sin}(X_{n-1}) \xrightarrow{j_{n-1}} H_{n-1}^{sin}(X_{n-1}, X_{n-2}),$$

where ∂ denotes the boundary operator of the topological pair (X_n, X_{n-1}) and j_{n-1} is the homomorphism induced by the inclusion $(X_{n-1}, \emptyset) \subset (X_{n-1}, X_{n-2})$. As is well known, the singular homology groups of a CW-complex are isomorphic to its cellular homology groups. The nth singular cohomology group $H^n_{sin}(X)$ with integral coefficients is given by the cohomology of the singular cochain complex $H^n(S^\bullet_{sin}(X))$, where $S^n_{sin}(X) = \operatorname{Hom}(S_n^{sin}(X), \mathbb{Z})$. If X has the structure of a CW-complex, then the singular cohomology of X is isomorphic to the cohomology of its cellular cochain complex

$$C^n_{sin}(X) = H^n_{sin}(X_n, X_{n-1}) \cong \prod\nolimits_{A_n} \mathbb{Z}.$$

1.2.2. Locally finite homology and compact support cohomology

A locally finite singular n-chain of a space X is a product $\prod_\alpha n_\alpha \sigma_\alpha$, with $n_\alpha \in \mathbb{Z}$ and $\sigma_\alpha : \Delta^n \to X$ singular n-simplexes, such that for each $x \in X$ there exists an open neighbourhood U of x such that $\{\alpha \mid U \cap \sigma_\alpha(\Delta) \neq \emptyset,\ n_\alpha \neq 0\}$ is finite. The locally finite singular chain complex $S_\bullet^{lf}(X)$ is the chain complex with $S_n^{lf}(X)$ the Abelian group of locally finite singular n-chains and the usual boundary homomorphisms. The locally finite homology of X is denoted by $H_n^{lf}(X)$.

A proper closed map $f : X \to Y$ between locally compact Hausdorff spaces induces, for each integer n, a homomorphism $f_* : H_n^{lf}(X) \to H_n^{lf}(Y)$.

The locally finite cellular chain complex of a strongly locally finite CW-complex X, $C_\bullet^{lf}(X)$, is defined by

$$C_n^{lf}(X) = H_n^{lf}(X_n, X_{n-1}) \cong \prod\nolimits_{A_n} \mathbb{Z},$$

and the boundary homomorphism d_n^{lf} is the composite

$$H_n^{lf}(X_n, X_{n-1}) \xrightarrow{\partial} H_{n-1}^{lf}(X_{n-1}) \xrightarrow{j_{n-1}} H_{n-1}^{lf}(X_{n-1}, X_{n-2}).$$

The chain complexes $S_\bullet^{lf}(X)$ and $C_\bullet^{lf}(X)$ are homology equivalent when X is strongly locally finite; that is, if X is the union of a countable, locally finite collection of finite subcomplexes. Note that every countable, locally finite, finite-dimensional CW-complex is a strongly locally finite CW-complex.

A q-cochain $c\colon S_q(X) \to \mathbb{Z}$ has *compact support* if there is compact subset $K \subset X$ such that if a q-simplex σ is contained in $X \setminus K$, then $c(\sigma) = 0$. The complex $S^\bullet_{cs}(X)$ of compact support cochains gives the nth compact support cohomology group $H^n_{cs}(X) = H^n(S^\bullet_{cs}(X))$.

The corresponding cellular cochain complex of a strongly locally finite CW-complex X, $C^\bullet_{cs}(X)$ is given by

$$C^n_{cs}(X) = H^n_{cs}(X_n, X_{n-1}) \cong \bigoplus_{A_n} \mathbb{Z}.$$

The cochain complexes $S^\bullet_{cs}(X)$ and $C^\bullet_{cs}(X)$ are homology equivalent when X is strongly locally finite. For more details and properties of these theories, we refer the reader to [9].

1.2.3. Homology and cohomology at infinity

The *singular chain complex at infinity* of a space X is given by $S^\infty_\bullet(X) = S^{lf}_\bullet(X)/S^{sin}_\bullet(X)$ and the corresponding nth singular homology group at infinity is denoted by $H^\infty_n(X)$.

If X is a strongly locally finite CW-complex, the *cellular chain complex at infinity* of X is given by

$$C^\infty_n(X) = H^\infty_n(X_n, X_{n-1}) \cong \prod_{A_n} \mathbb{Z} / \bigoplus_{A_n} \mathbb{Z}.$$

When X is a strongly locally finite CW-complex, we have that the singular locally finite homology is given by the cellular chain complex at infinity.

The *singular cochain complex at infinity* of a space X is given by $S^n_\infty(X) = S^\bullet_{sin}(X)/S^\bullet_{cs}(X)$ and the corresponding nth singular cohomology group at infinity is denoted by $H^n_\infty(X)$.

If X is a strongly locally finite CW-complex, the *cellular cochain complex at infinity* of X is given by

$$C^n_\infty(X) = H^n_\infty(X_n, X_{n-1}) \cong \prod_{A_n} \mathbb{Z} / \bigoplus_{A_n} \mathbb{Z}.$$

When X is a strongly locally finite CW-complex, we have that the compact support cohomology is isomorphic to the cohomology of the cellular cochain complex at infinity. We also refer to these homology and cohomology groups as the end homology and end cohomology, see [9].

1.2.4. Čech homology and cohomology

If X is a compact metric space, let $\mathcal{U}_i$ be a sequence of finite open covers of X such that $\mathcal{U}_{i+1}$ refines $\mathcal{U}_i$ and $\lim_{i\to\infty} \sup\{\operatorname{diam}(U)/U \in \mathcal{U}_i\} = 0$.

The Čech homology of X is given by the inverse limit $\check{H}_n(X) = \lim\{H^{sin}_n(N_i)\}$, where N_i denotes the (Čech) nerve of $\mathcal{U}_i$. The Čech cohomology is defined to be the direct limit $\check{H}^n(X) = \lim\{H^n_{sin}(N_i)\}$. For a formulation of Čech homology and cohomology groups based on more general resolutions we refer the reader to [11].

1.2.5. Steenrod homology

Let X be a compact metric space. A *regular map* of a simplicial complex K in X is a function f defined over the vertices of K with values in X, such that, for all $\varepsilon > 0$, all but finitely many simplexes have their vertices imaging onto sets of diameter $< \varepsilon$. A *regular n-chain* of X is a triple (K, f, σ^n), where K is a simplicial complex, f is a regular map of K in X and σ^n is a locally finite n-chain of K. If σ^n is an n-cycle, then (K, f, σ^n) is a regular n-cycle. Two regular n-cycles (K_1, f_1, σ_1^n), (K_2, f_2, σ_2^n) are *homologous* if there exists an $(n+1)$-chain (K, f, σ^{n+1}) such that K_1 and K_2 are closed subcomplexes of K, f agrees with f_1 on K_1 and f_2 on K_2 and $\partial(\sigma^{n+1}) = \sigma_1^n - \sigma_2^n$. This construction was introduced by Steenrod [16] and the corresponding reduced homology of a compact metric space X will be denoted by $\widetilde{H}_n^{St}(X)$. For more properties of Steenrod homology we refer the reader to [7,12,2,4].

2. Axioms of homology theory for exterior spaces. Cellular homology

In this section we analyze a family of axioms for a homology theory on the category $\boldsymbol{E}^{(\mathbf{2})}$ of exterior pairs and maps and the induced cellular homology for exterior spaces which admit the structure of a bi-complex.

By an *exterior pair* (X, A) we mean an exterior space X and a subspace $A \subset X$ provided with ε_A, which consists of those open subsets of the form $E \cap A$, where $E \in \varepsilon_X$. In the obvious way, we have the notion of an exterior map $f : (X, A) \to (Y, B)$ between exterior pairs.

Given two exterior maps $f_0, f_1 : (X, A) \to (Y, B)$, f_0 and f_1 are said to be *homotopic* if there exists an exterior map $h : (X \bar{\times} I, A \bar{\times} I) \to (Y, B)$ such that $h(x, 0) = f_0(x)$ and $h(x, 1) = f_1(x)$, for all $x \in X$.

Definition 2.1. An ordinary homology theory on $\boldsymbol{E}^{(\mathbf{2})}$ with 'values' in an Abelian category $\mathcal{A}$ consists of a collection $\mathfrak{H} = (H, \partial)$ where H is a sequence of functors $H_q : \boldsymbol{E}^{(\mathbf{2})} \to \mathcal{A}$, indexed by the set of all integers, and ∂ is a sequence of natural transformations $\partial_q = \partial : H_q(X, A) \to H_{q-1}(A)$, called boundary operators, such that $\mathfrak{H}$ satisfies the following axioms:

- (A1) (*Exactness axiom*) Let (X, A) be an exterior pair and $i : A \to X$, $j : X \to (X, A)$ denote the inclusion maps. Then, the sequence
$$\cdots \to H_q(A) \xrightarrow{i_*} H_q(X) \xrightarrow{j_*} H_q(X, A) \xrightarrow{\partial} H_{q-1}(A) \to \cdots$$
is exact, where i_*, j_* denote $H_q(i)$ and $H_q(j)$, respectively.
- (A2) (*Homotopy invariance axiom*) If $f, g : (X, A) \to (Y, B)$ are homotopic exterior maps, then $f_* = g_* : H_q(X, A) \to H_q(Y, B)$, for every integer q.
- (A3) (*Excision axiom*) Let X be an exterior space which is first countable at infinity and U be an open subset of X such that $Cl(U) \subset Int(A)$, where A is an exterior subspace of X. Then, the inclusion map $i : (X - U, A - U) \to (X, A)$ induces an isomorphism $i_* : H_q(X - U, A - U) \to H_q(X, A)$, for every integer q.
- (A4) (*Dimension axiom*) $H_q(\mathbb{N}) = 0$ for every integer $q \neq 0$.

The object $G = H_0(\mathbb{N})$ is called the *coefficient object* of the homology theory $\mathfrak{H}$.

Definition 2.2. A cohomology theory on $\boldsymbol{E}^{(\mathbf{2})}$ with values in an Abelian category $\mathcal{A}$ is a homology theory with values in the opposite category $\mathcal{A}^{\mathbf{op}}$.

It is easy to check that for each exterior pair (X, A), where X is an exterior space which is first countable at infinity and X is the disjoint union of n open subspaces $X_1, X_2, \ldots, X_n$, the inclusion map $i_k : (X_k, A_k) \to (X, A)$, with $A_k = A \cap X_k$, induces a monomorphism $(i_k)_* : H_q(X_k, A_k) \to H_q(X, A)$, for $k = 1, 2, \ldots, n$, and every integer q. Furthermore, the homomorphism

$$\phi = \sum_{k=1}^{n} (i_k)_* : \bigoplus_{k=1}^{n} H_q(X_k, A_k) \to H_q(X, A)$$

is an isomorphism for every integer q.

As a consequence, we have that, in general, it is not possible to find a homology theory on $\boldsymbol{E}^{(\mathbf{2})}$ associated with every object G. The isomorphism $\mathbb{N} \sqcup \mathbb{N} \cong \mathbb{N}$ gives rise to an induced isomorphism $G \oplus G \cong G$, and therefore the coefficient object G has this nice property. Several examples of ordinary homology theories on $\boldsymbol{E}^{(\mathbf{2})}$ will be developed in next section.

Throughout this section it is assumed that $\mathfrak{H} = (H, \partial)$ is an arbitrarily given ordinary homology theory on $\boldsymbol{E}^{(\mathbf{2})}$.

If P denotes the singleton set, we will consider the exterior space $(P, \{\emptyset, P\} \subset \{\emptyset, P\})$. The following result establishes that the homology of P is determined by the homology of $\mathbb{N}$.

Proposition 2.3. *Let* $sh : \mathbb{N} \to \mathbb{N}$ *be the 'shift operator' given by* $sh(k) = k + 1$, *for all* $k \in \mathbb{N}$. *Then*

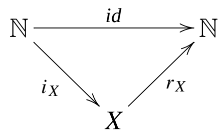
$$H_n(P) = \begin{cases} \operatorname{coker}((sh)_* : H_0(\mathbb{N}) \to H_0(\mathbb{N})) & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We consider the homology sequence of $(\mathbb{N}, \mathbb{N}(1))$, where $\mathbb{N}(1) = \{k \in \mathbb{N}:\ k \geqslant 1\}$. Taking into account that $sh' : \mathbb{N} \to \mathbb{N}(1)$, defined by $sh'(k) = k + 1$, is an isomorphism in $\boldsymbol{E}$ and $i(sh') = sh$, where i denotes the inclusion $\mathbb{N}(1) \subset \mathbb{N}$, we obtain an exact sequence

$$0 \to H_1\big(\mathbb{N}, \mathbb{N}(1)\big) \to H_0(\mathbb{N}) \xrightarrow{(sh)_*} H_0(\mathbb{N}) \to H_0\big(\mathbb{N}, \mathbb{N}(1)\big) \to 0.$$

Since the map $r : \mathbb{N} \to \mathbb{N}$ given by $r(k) = k - 1$ if $k \geqslant 1$ and $r(0) = 0$ satisfies that $r\, sh = id$, then $(sh)_*$ is a monomorphism, so $H_1(\mathbb{N}, \mathbb{N}(1)) = 0$. Note that by using the excision axiom we have that $H_q(\mathbb{N}, \mathbb{N}(1)) = H_q(P)$. We conclude that $H_0(P) \cong \operatorname{coker}((sh)_*)$. The rest of the proof is straightforward. $\square$

We introduce what we call the $\mathbb{N}$-*reduced homology* associated with $\mathfrak{H}$ on the category $\overline{\boldsymbol{E}}{}^{\mathbb{N}}_{\mathbb{N}}$ whose objects are commutative triangles in $\boldsymbol{E}$ of the form



denoted by (i_X, X, r_X). A morphism $(i_X, X, r_X) \to (i_Y, Y, r_Y)$ consists of an exterior map $f : X \to Y$ such that $f i_X = i_Y$ and $r_Y f = r_X$. Observe that $\mathfrak{S}^n$ can be considered as an object in $\overline{\boldsymbol{E}}_{\mathbb{N}}^{\mathbb{N}}$, taking $i_{\mathfrak{S}^n}(k) = (k, *)$ and $r_{\mathfrak{S}^n}(k, x) = k$, for all $k \in \mathbb{N}$ and $x \in S^n$. Here $*$ denotes the base point of S^n, $(1, 0, \ldots, 0)$.

Another special object is $\mathbb{N}$, with $i_{\mathbb{N}} = r_{\mathbb{N}} = id_{\mathbb{N}}$.

Definition 2.4. Let (i_X, X, r_X) be an object of $\overline{\boldsymbol{E}}_{\mathbb{N}}^{\mathbb{N}}$. We define its qth $\mathbb{N}$-*reduced homology* as the kernel

$$\widetilde{H}_q(X) = \ker\bigl((r_X)_* : H_q(X) \to H_q(\mathbb{N})\bigr).$$

Obviously this defines a functor $\overline{\boldsymbol{E}}_{\mathbb{N}}^{\mathbb{N}} \to \mathcal{A}$ for every integer q. The $\mathbb{N}$-reduced homology has nice properties, such as the existence of an exact sequence of the $\mathbb{N}$-reduced homology associated with an inclusion $A \subset X$ in $\overline{\boldsymbol{E}}_{\mathbb{N}}^{\mathbb{N}}$ and the homotopy invariance. Furthermore, there is an isomorphism $H_q(X) \cong H_q(\mathbb{N}) \oplus \widetilde{H}_q(X)$, for every integer q.

Using analogous techniques to those considered in singular homology theory, it is not difficult to see that $\widetilde{H}_q(\mathfrak{S}^n)$ is G if $q = n$ and 0 if $q \neq n$. Hence we have that

$$H_q(\mathfrak{S}^n) = \begin{cases} 0 & \text{if } n \neq q \neq 0, \\ G & \text{if } n \neq q = 0 \text{ or } n = q \neq 0, \\ G \oplus G & \text{if } n = q = 0. \end{cases}$$

On the other hand, in order to compute $H_q(S^n)$, where S^n has the topology as the collection of e-open subsets, we introduce the P-reduced homology associated with $\mathfrak{H}$ on the category $\boldsymbol{E}_{\boldsymbol{P}}^{\boldsymbol{P}}$ of exterior spaces over and under P. We define

$$\widetilde{\widetilde{H}}(X) = \ker\bigl((r_X)_* : H_q(X) \to H_q(P)\bigr)$$

for each exterior space over and under P and integer q. By similar arguments we have that $H_q(S^n)$ is 0 if $n \neq q \neq 0$, G' if $n \neq q = 0$ or $n = q \neq 0$, and $G' \oplus G'$ if $n = q = 0$, where G' denotes $H_0(P)$.

Definition 2.5. By a *finite bi-complex pair* we mean an exterior pair (X, A), where X is a bi-complex with a finite number of cells and A is a subcomplex of X.

Let (X, A) be a finite bi-complex pair such that X is obtained from A by attaching n-cells

$$\begin{array}{ccc} (\coprod_{i=1}^{k} \mathfrak{S}_i^{n-1}) \sqcup (\coprod_{i=k+1}^{m} S_i^{n-1}) & \xrightarrow{(\coprod_{i=1}^{k} \varphi_i) \sqcup (\coprod_{i=k+1}^{m} \varphi_i)} & A \\ \downarrow & & \downarrow{\scriptstyle i_n} \\ (\coprod_{i=1}^{k} \mathfrak{D}_i^{n}) \sqcup (\coprod_{i=k+1}^{m} D_i^{n}) & \xrightarrow[(\coprod_{i=1}^{k} \psi_i) \sqcup (\coprod_{i=k+1}^{m} \psi_i)]{} & X \end{array}$$

Using the excision property, one can prove that

$$(\psi_i)_* : H_q\bigl(\mathfrak{D}_i^n, \mathfrak{S}_i^{n-1}\bigr) \to H_q(X, A) \quad (1 \leqslant i \leqslant k)$$

and

$$(\psi_i)_*: H_q\big(D_i^n, S_i^{n-1}\big) \to H_q(X, A) \quad (k+1 \leqslant i \leqslant m)$$

are monomorphisms and the homomorphism

$$h = \sum_{i=1}^{m} (\psi_i)_*: H_q\big(\mathfrak{D}_i^n, \mathfrak{S}_i^{n-1}\big)^k \oplus H_q\big(D_i^n, S_i^{n-1}\big)^{m-k} \to H_q(X, A)$$

is an isomorphism, where the exponents denote the number of copies of the corresponding object. Since we have that $H_q(\mathfrak{D}^n, \mathfrak{S}^{n-1})$ is G if $q = n$ and 0 if $q \neq n$, and $H_q(D_i^n, S_i^{n-1})$ is G' if $q = n$ and 0 if $q \neq n$, we have as a consequence that

$$H_q(X, A) = \begin{cases} G^k \oplus (G')^{m-k} & \text{if } q = n, \\ 0 & \text{otherwise.} \end{cases}$$

By the *relative dimension* of a finite bi-complex pair (X, A), $\dim(X, A)$, we mean the smallest integer n satisfying $X - A \subset X_n$. In case $X - A = \emptyset$ we set $\dim(X, A) = -1$. By using inductive arguments, we can prove without difficulty that if $\dim(X, A) = n$, then $H_q(X, A) = 0$ for every $q < 0$ and $q > n$. In particular, given a finite bi-complex X, then $H_q(X_n) = 0$ for all $q > n$.

Definition 2.6. The *cellular chain complex* $C_\bullet(X)$ of a bi-complex X, is defined to be

$$C_n(X) = H_n(X_n, X_{n-1}),$$

with boundary homomorphism d_n the composite

$$H_n(X_n, X_{n-1}) \xrightarrow{\partial} H_{n-1}(X_{n-1}) \xrightarrow{j_{n-1}} H_{n-1}(X_{n-1}, X_{n-2}),$$

where ∂ denotes the boundary operator of the exterior pair (X_n, X_{n-1}) and j_{n-1} is the homomorphism induced by the inclusion $(X_{n-1}, \emptyset) \subset (X_{n-1}, X_{n-2})$.

The nth *cellular homology* of X, $H_n^{cel}(X)$, is the nth homology of this chain complex. We are going to prove that the cellular homology and the homology coincide on finite bi-complexes. In fact, we shall give an algorithm which allows us to compute the homology for this distinguished class of exterior spaces.

We consider the diagram

$$H_n(X) \xleftarrow{k_n} H_n(X_n) \xrightarrow{j_n} H_n(X_n, X_{n-1}),$$

where k_n and j_n are the homomorphisms induced by the respective inclusion maps.

Theorem 2.7. *Let X be a finite bi-complex. Then*

(a) *k_n is an epimorphism,*

(b) *j_n is a monomorphism, and*

(c) $\operatorname{im}(j_n) = \ker(d_n)$, $\ker(k_n) = j_n^{-1}(\operatorname{im}(d_n))$.

Hence, $\theta_n = j_n k_n^{-1}: H_n(X) \to H_n^{cel}(X)$ is an isomorphism.

Proof. Consider the following part of the exact homology sequence associated to the exterior pair (X_n, X_{n-1}):

$$\begin{aligned} 0 \longrightarrow H_n(X_n) &\xrightarrow{j_n} H_n(X_n, X_{n-1}) \xrightarrow{\partial} H_{n-1}(X_{n-1}) \\ &\xrightarrow{i_n} H_{n-1}(X_n) \longrightarrow 0. \end{aligned} \tag{$*$}$$

If $q \neq n$ and $q \neq n-1$, then $H_{q+1}(X_n, X_{n-1}) = 0 = H_q(X_n, X_{n-1})$, so $i_n : H_q(X_{n-1}) \to H_q(X_n)$ is an isomorphism.

Suppose that $\dim(X) \leqslant m$ and $n \leqslant m$. If $q < m$, we have the following commutative diagram:

$$\begin{array}{ccccccccc} H_q(X_{q+1}) & \xrightarrow[\cong]{i_{q+2}} & H_q(X_{q+2}) & \xrightarrow[\cong]{i_{q+3}} & H_q(X_{q+3}) & \xrightarrow[\cong]{i_{q+4}} & \cdots & \xrightarrow[\cong]{i_m} & H_q(X_m) \\ & \searrow^{k_{q+1}} & \downarrow{\scriptstyle k_{q+2}} & \swarrow_{k_{q+3}} & & & {\scriptstyle k_m = id} & & \swarrow \\ & & H_q(X) & & & & & & \end{array}$$

It follows that $k_\alpha : H_q(X_\alpha) \to H_q(X)$ is an isomorphism, for all $\alpha > q$. Hence, $k_{q+1} : H_q(X_{q+1}) \to H_q(X)$ is an isomorphism. From $(*)$ we deduce that j_n is a monomorphism and i_n an epimorphism. Since $k_n i_n = k_{n-1}$ and k_n is an isomorphism, it follows that k_{n-1} is an epimorphism. Furthermore, $\ker(k_{n-1}) = \ker(i_n)$, so the following sequence is exact:

$$0 \longrightarrow H_n(X_n) \xrightarrow{j_n} H_n(X_n, X_{n-1}) \xrightarrow{\partial} H_{n-1}(X_{n-1}) \xrightarrow{k_{n-1}} H_{n-1}(X) \longrightarrow 0.$$

Since $d_n = j_{n-1}\partial$ and j_{n-1} is a monomorphism, then we have that $\ker(d_n) = \ker(\partial) = \operatorname{im}(j_n)$. On the other hand $\ker(k_{n-1}) = \operatorname{im}(\partial) = j_{n-1}^{-1}(\operatorname{im}(j_{n-1}\partial)) = j_{n-1}^{-1}(\operatorname{im}(d_n))$. □

One can prove that the same result holds for finite bi-complex pairs (X, A), using the chain complex $C_n(X, A) = H_n(K_n, K_{n-1})$, where $K_n = X_n \cup A$. On the other hand, if H_q satisfies the condition that $H_q(X) = \operatorname{colim} H_q(X_n)$, then the theorem also holds for bi-complexes with a finite number of cells in each dimension.

Finally, note that finite $\mathbb{N}$-complexes and finite CW-complexes embedded in $\boldsymbol{E}$ are finite bi-complexes.

Remark 2.8. Note that we have similar results for cohomology theories with the cochain complex $C^n(X) = H^n(K_n, K_{n-1})$.

3. Examples of homology theories on $\boldsymbol{E}^{(2)}$

3.1. Sequential homology

By a *locally finite matrix* we mean an infinite integer matrix with rows and columns indexed by elements of $\mathbb{N}$ such that each row and each column contains only a finite number of nonzero entries (see [6]). We will denote the ring of locally finite matrices by $\mathcal{R}$. We can also consider $\mathcal{R}$ as an endomorphism ring in the category of external Abelian groups.

Definition 3.1. Let G be an Abelian group. An *externology* on G is a nonempty collection ε of subgroups of G satisfying:

(E1) if $E_1, E_2 \in \varepsilon$ then $E_1 \cap E_2 \in \varepsilon$,

(E2) if $E \in \varepsilon$, $U < G$ and $E < U$ then $U \in \varepsilon$.

An *external Abelian group* (G, ε) consists of an Abelian group G together with an externology ε. A subgroup E in ε is said to be an *exterior subgroup* or *e-subgroup*.

A homomorphism $f:(G,\varepsilon) \to (G',\varepsilon')$ is said to be *external* if $f^{-1}(E) \in \varepsilon$ for all $E \in \varepsilon'$.

The category of external Abelian groups and homomorphisms will be denoted by **e-Ab**. One can check that **e-Ab** is an additive complete and cocomplete category.

Definition 3.2. Given an external Abelian group (G, ε), an *external base* is a collection of e-subgroups, β, such that for every e-subgroup E there exists $B \in \beta$ such that $B < E$.

We will use the Abelian group $\mathfrak{Z} = \bigoplus_0^\infty \mathbb{Z}$ provided with the externology consisting of those subgroups $E < \mathfrak{Z}$ such that there exists $i \in \mathbb{N}$ such that $\bigoplus_i^\infty \mathbb{Z} < E$. Hence $\{\bigoplus_i^\infty \mathbb{Z}\}_{i=0}^\infty$ constitutes an external base for $\bigoplus_0^\infty \mathbb{Z}$.

The elements $e_0 = (1, 0, \ldots)$, $e_1 = (0, 1, 0, \ldots), \ldots$ clearly generate $\mathfrak{Z}$. Taking into account that an external homomorphism $a : \mathfrak{Z} \to G$ is determined by the images $\{a(e_i)\}_{i=0}^\infty$, one can prove that $\mathrm{Hom}_{\mathbf{e\text{-}Ab}}(\mathfrak{Z}, \mathfrak{Z})$, with the obvious sum and the composition, and $\mathcal{R}$ are isomorphic as rings.

Let X be an exterior space. We consider the positive chain complex of external Abelian groups, $S_\bullet^{sin}(X)$, where $S_n^{sin}(X)$ is the Abelian group of all singular n-chains on X, provided with the externology whose base is $\{S_n^{sin}(E)$: E is an e-open subset of $X\}$. One can check that the singular boundary homomorphism $d_n^{sin} : S_n^{sin}(X) \to S_{n-1}^{sin}(X)$ is external, for each $n \geqslant 0$.

Definition 3.3. Let X be an exterior space. We define the *sequential chain complex* of $\mathcal{R}$-modules, $S_\bullet^{seq}(X)$, by

$$S_n^{seq}(X) = \mathrm{Hom}_{\mathbf{e\text{-}Ab}}\big(\mathfrak{Z}, S_n^{sin}(X)\big), \quad d_n^{seq} = \big(d_n^{sin}\big)_*.$$

Given (X, A) an exterior pair, the chain complex $S_\bullet^{seq}(X, A)$ is defined by $S_\bullet^{seq}(X)/S_\bullet^{seq}(A)$. The nth homology $\mathcal{R}$-module of this chain complex is denoted by $H_n^{seq}(X, A)$, and it will be called the nth *sequential homology* of (X, A). This construction clearly defines, for each integer n, a functor $H_n^{seq} : \boldsymbol{E}^{(2)} \to \mathcal{R}$-**Mod**, where $\mathcal{R}$-**Mod** denotes the category of $\mathcal{R}$-modules and homomorphisms of $\mathcal{R}$-modules. The connecting homomorphism of the exact sequence

$$0 \to S_\bullet^{seq}(A) \to S_\bullet^{seq}(X) \to S_\bullet^{seq}(X, A) \to 0$$

gives rise to the boundary operator $\partial^{seq} : H_n^{seq}(X, A) \to H_{n-1}^{seq}(A)$.

Theorem 3.4. *$\mathfrak{H} = (H^{seq}, \partial^{seq})$ is a homology theory on $\boldsymbol{E}^{(2)}$.*

Proof. Axiom (A1) is immediately satisfied. In order to prove axiom (A2), let $F:(X\bar{\times}I, A\bar{\times}I)\to(Y,B)$ be a homotopy between f and g. We consider

$$\Gamma_n^{(X,A)}:S_n^{sin}(X,A)\to S_{n+1}^{sin}(X\bar{\times}I, A\bar{\times}I)$$

the prism operator [5] and

$$L_n=S_{n+1}^{sin}(F)\Gamma_n^{(X,A)}:S_n^{sin}(X,A)\to S_{n+1}^{sin}(Y,B).$$

It is straightforward to see that L_n is an external homomorphism. Furthermore, $L=\{(L_n)_*:S_n^{seq}(X,A)\to S_{n+1}^{seq}(Y,B)\}_{n=0}^{\infty}$ is a chain homotopy $L:S_\bullet^{seq}(f)\simeq S_\bullet^{seq}(g)$.

Note that

$$\mathrm{Hom}_{\mathbf{e\text{-}Ab}}\big(\mathfrak{Z}, S_\bullet^{sin}(X,A)\big)\cong S_\bullet^{seq}(X,A).$$

Now, let (X,A) be an exterior pair, where X is an exterior space first countable at infinity, and suppose U is an open subset of X such that $Cl_X(U)\subset Int_X(A)$. We consider $X=E_0\supset E_1\supset E_2\supset\cdots\supset E_n\supset\cdots$ an exterior base for X. Taking into account that $Cl_{E_k}(E_k\cap U)\subset Int_{E_k}(E_k\cap A)$, for each k we have that the inclusion maps

$$i_{E_k}:\big(E_k-(E_k\cap U),(E_k\cap A)-(E_k\cap U)\big)\to(E_k,E_k\cap A)$$

induce isomorphisms on the singular homology groups

$$(i_{E_k})_*:H_n^{sin}\big(E_k-(E_k\cap U),(E_k\cap A)-(E_k\cap U)\big)\to H_n^{sin}(E_k,E_k\cap A).$$

$\{S_n^{sin}(E_k)\}_{k=0}^{\infty}$ and $\{S_n^{sin}(E_k,E_k\cap A)\}_{k=0}^{\infty}$ are external bases for $S_n^{sin}(X)$ and $S_n^{sin}(X,A)$, respectively, so if we take $[\sigma]\in H_n^{seq}(X,A)$ represented by the external homomorphism $\sigma:\mathfrak{Z}\to S_n^{sin}(X,A)$, there is an increasing monotone sequence $\{\varphi(i)\}_{i=0}^{\infty}\subset\mathbb{N}$ such that for each $l\in\mathbb{N}$ and $k\geqslant\varphi(l)$, $\sigma(e_k)\in S_n^{sin}(E_l,E_l\cap A)$. If $\varphi(l)\leqslant k<\varphi(l+1)$, then $[\sigma(e_k)]\in H_n^{sin}(E_l,E_l\cap A)$ because $(d_n^{sin})_*(\sigma)=0$ implies $d_n^{sin}(\sigma(e_k))=0$. Since $(i_{E_l})_*$ is an isomorphism, we take $[\tilde{\sigma}(e_k)]\in H_n^{sin}(E_l-(E_l\cap U),(E_l\cap A)-(E_l\cap U))$ verifying $(i_{E_l})_*([\tilde{\sigma}(e_k)])=[\sigma(e_k)]$. It is easy to check that this argument gives us an element $[\tilde{\sigma}]\in H_n^{seq}(X-U,A-U)$ such that $i_*([\tilde{\sigma}])=[\sigma]$, where i denotes the inclusion $(X-U,A-U)\subset(X,A)$. Hence $H_n^{seq}(i)=i_*$ is an epimorphism. By similar arguments i_* is a monomorphism, so (A3) is satisfied.

In order to prove axiom (A4) we will see that

$$H_n^{seq}(\mathbb{N})=\begin{cases}\mathcal{R}, & \text{if } n=0,\\ 0, & \text{otherwise.}\end{cases}$$

Since $\{\mathbb{N}(i)\}_{i=0}^{\infty}$ is an exterior base for $\mathbb{N}$, with $\mathbb{N}(i)=\{k\in\mathbb{N}:\ k\geqslant i\}$. Then $\{S_n^{sin}(\mathbb{N}(i))\}_{i=0}^{\infty}$ is an external base for $S_n(\mathbb{N})$, $n\geqslant 0$. Then, using techniques similar to those used to prove (A3) and taking into account that $H_n^{sin}(\mathbb{N}(i))=0$ if $n\neq 0$, we have that $H_n^{seq}(\mathbb{N})=0$ when $n\neq 0$. Now we analyze the case $H_0^{seq}(\mathbb{N})$. Since $d_1=0$, then $H_0^{seq}(\mathbb{N})=\mathrm{Hom}_{\mathbf{e\text{-}Ab}}(\mathfrak{Z},S_0^{sin}(\mathbb{N}))$, but $S_0^{sin}(\mathbb{N})=\mathfrak{Z}$ so $H_0^{seq}(\mathbb{N})=\mathrm{Hom}_{\mathbf{e\text{-}Ab}}(\mathfrak{Z},\mathfrak{Z})=\mathcal{R}$. □

Singular homology and sequential homology are related to each other.

Proposition 3.5. *There are natural isomorphisms, where X is in the category of topological spaces*:

(a) $H_n^{seq}(X) \cong \prod_0^\infty H_n^{sin}(X)$, *if* $\varepsilon_X = \{X\}$, *and*

(b) $H_n^{seq}(X) \cong \bigoplus_0^\infty H_n^{sin}(X)$, *if* ε *is the topology of* X.

Proof. (a) In this case the externology on $S_n^{sin}(X)$ is $\{S_n^{sin}(X)\}$. Hence

$$\mathrm{Hom}_{\mathbf{e\text{-}Ab}}\big(\mathfrak{Z}, S_n^{sin}(X)\big) = \mathrm{Hom}_{\mathbf{Ab}}\big(\mathfrak{Z}, S_n^{sin}(X)\big) \cong \prod_0^\infty \mathrm{Hom}_{\mathbf{Ab}}\big(\mathbb{Z}, S_n^{sin}(X)\big) \cong \prod_0^\infty S_n^{sin}(X).$$

Therefore $H_n^{seq}(X) \cong \prod_0^\infty H_n^{sin}(X)$.

(b) Since $\{\emptyset\}$ is an exterior base for X, then $\{0\}$ is a external base for $S_n^{sin}(X)$. For each external homomorphism $a : \mathfrak{Z} \to S_n(X)$, there is a nonnegative integer k_a such that $a_k = a(e_k) = 0$ for all $k \geqslant k_a$. Then $S_n^{seq}(X) = \bigoplus_0^\infty S_n^{sin}(X)$ with $d_n^{seq} = \bigoplus_0^\infty d_n^{sin}$. □

As a consequence, for each compact space X, we have that

$$H_n^{seq}(X_e) \cong \bigoplus_0^\infty H_n^{sin}(X).$$

Remark 3.6. The proper homotopy groups introduced by Brown [1] have global versions, see [8], and one can consider Hurewicz maps from 'global Brown homotopy groups' to sequential homology groups.

3.2. Tubular homology

Let X be an exterior space. The tubular chain complex of X, $S_\bullet^{tub}(X)$, is defined by the following chain complex of Abelian groups:

$$S_n^{tub}(X) = S_{n-1}^{seq}(X) \oplus S_n^{seq}(X);$$
$$d_n^{tub}(a, x) = \big(d_{n-1}^{seq}(a), -d_{n-1}^{seq}(x) + a - aSh\big),$$

where Sh denotes the locally finite matrix defined by $Sh(e_k) = e_{k+1},\ k = 0, 1, \ldots.$

For exterior maps $f : X \to Y$, $S_n^{tub}(f) = S_{n-1}^{seq}(f) \oplus S_n^{seq}(f)$. The nth *tubular homology group* of X is $H_n(S_\bullet^{tub}(X))$. It will be denoted by $H_n^{tub}(X)$. For exterior pairs (X, A) we consider $S_n^{tub}(X, A) = S_{n-1}^{seq}(X, A) \oplus S_n^{seq}(X, A)$ with the obvious boundary homomorphisms, and $H_n^{tub}(X, A) = H_n(S_\bullet^{tub}(X, A))$. It is easy to check that $S_\bullet^{tub}(X, A) \cong S_\bullet^{tub}(X)/S_\bullet^{tub}(A)$.

Clearly this defines a functor $H_n^{tub} : \boldsymbol{E}^{(2)} \to \mathbf{Ab}$ for each n. By similar arguments to those used for H^{seq}, one can prove without difficulty axioms (A1), (A2) and (A3). In order to see axiom (A4) we need some previous results. First we will see a relation between tubular homology and singular homology in terms of the inverse limit and its first derived functor associated with certain inverse systems of Abelian groups. Recall that if

$$A_0 \xleftarrow{p_0} A_1 \xleftarrow{p_1} A_2 \xleftarrow{p_2} \cdots$$

is an inverse system of Abelian groups and if $d:\prod_{i=0}^{\infty} A_i \to \prod_{i=0}^{\infty} A_i$ is the homomorphism defined by $d(a_0, a_1, a_2, \ldots) = (a_0 - p_0(a_1), a_1 - p_1(a_2), a_2 - p_2(a_3), \ldots)$, then $\ker(d) = \lim\{A_i\}$ and $\operatorname{coker}(d) = \lim^1\{A_i\}$. Two important properties of the derived functor $\lim^1$, see [3,9], are the following:

– If each projection p_i is an epimorphism, then $\lim^1\{A_i\} = 0$.
– Every short exact sequence of inverse systems,

$$0 \longrightarrow \{A_i\} \longrightarrow \{B_i\} \longrightarrow \{C_i\} \longrightarrow 0,$$

gives rise to an exact sequence of Abelian groups

$$\begin{aligned} 0 &\longrightarrow \lim\{A_i\} \longrightarrow \lim\{B_i\} \longrightarrow \lim\{C_i\} \\ &\longrightarrow \lim^1\{A_i\} \longrightarrow \lim^1\{B_i\} \longrightarrow \lim^1\{C_i\} \longrightarrow 0. \end{aligned}$$

If X is an exterior space which is first countable at infinity and $X = E_0 \supset E_1 \supset E_2 \supset \cdots \supset E_n \supset \cdots$ a fixed countable exterior base, the inclusions $E_{i+1} \subset E_i$ give rise to the induced homomorphism $p_i : H_q^{sin}(E_{i+1}) \to H_q^{sin}(E_i)$, $q \geqslant 0$ and the inverse system of Abelian groups $\{H_q^{sin}(E_i)\}$.

Proposition 3.7. *There is a short exact sequence*

$$0 \longrightarrow \lim^1\big\{H_q^{sin}(E_i)\big\} \xrightarrow{\alpha} H_q^{tub}(X) \xrightarrow{\beta} \lim\big\{H_{q-1}^{sin}(E_i)\big\} \longrightarrow 0.$$

Proof. α is defined as follows. We consider $\varphi:\prod_{i=0}^{\infty} H_q^{sin}(E_i) \to H_q^{tub}(X)$ defined by $\varphi(\{[x_i]\}_{i=0}^{\infty}) = [(0, x)]$, where $x:\mathfrak{Z} \to S_q(X)$ is given by $x(e_i) = x_i$. Since $\varphi d = 0$ we have that φ induces a homomorphism $\alpha:\lim^1\{H_q^{sin}(E_i)\} \to H_q^{tub}(X)$ such that $\alpha\pi = \varphi$, where π denotes the canonical projection.

On the other hand, for each q-cycle of $S_\bullet^{tub}(X)$, (a, x), there exists a monotone increasing sequence $\{n_i\}_{i=0}^{\infty} \subset \mathbb{N}$, with $n_0 = 0$ such that $(a_k, x_k) \in S_{q-1}^{sin}(E_i) \oplus S_q^{sin}(E_i)$, for all $k \geqslant n_i$; here a_k and x_k denote $a(e_k)$ and $x(e_k)$, respectively. We define $\beta([(a, x)]) = \{[a_{n_i}]\}_{i=0}^{\infty} \in \lim\{H_{q-1}^{sin}(E_i)\}$.

The facts that α is a monomorphism, β an epimorphism and $\operatorname{im}(\alpha) = \ker(\beta)$ are routine and are left as an exercise. □

Remark 3.8. There is a similar exact sequence for the relative case. If (X, A) is an exterior pair and $X = E_0 \supset E_1 \supset E_2 \supset \cdots \supset E_n \supset \cdots$ a fixed countable exterior base of X, there exists a short exact sequence:

$$\begin{aligned} 0 &\longrightarrow \lim^1\big\{H_q^{sin}(E_i, E_i \cap A)\big\} \xrightarrow{\alpha} H_q^{tub}(X, A) \\ &\xrightarrow{\beta} \lim\big\{H_{q-1}^{sin}(E_i, E_i \cap A)\big\} \longrightarrow 0 \end{aligned}$$

for every integer q.

As a consequence of this proposition we can compute the tubular homology of $\mathbb{N}$, considering the exterior base $\{\mathbb{N}(i)\}_{i=0}^{\infty}$, where $\mathbb{N}(i) = \{k \in \mathbb{N}\colon\ k \geqslant i\}$. The details are left to the reader.

Corollary 3.9.

$$H_q^{tub}(\mathbb{N}) = \begin{cases} \prod_0^\infty \mathbb{Z}/\bigoplus_0^\infty \mathbb{Z}, & \textit{if } q = 0, \\ 0, & \textit{otherwise.} \end{cases}$$

Proposition 3.10. *Let X be a topological space provided with the externology of all open subsets. Then $H_q^{tub}(X) = 0$, for all q.*

Proof. In this case,

$$S_n^{tub}(X) = \left(\bigoplus_0^\infty S_{n-1}^{sin}(X)\right) \oplus \left(\bigoplus_0^\infty S_n^{sin}(X)\right)$$

with boundary homomorphism

$$d_n^{tub}\big(\{a_i\},\{x_i\}\big) = \big(\{d_{n-1}^{sin}(a_i)\},\{-d_n^{sin}(x_i) + a_i - a_{i+1}\}\big).$$

It is not difficult to see that each q-cycle in $S_\bullet^{tub}(X)$ is a q-boundary. □

In particular, if X is a compact space (for example, the singleton space P), then $H_q^{tub}(X_e) = 0$, for all q.

The sequential homology and the tubular homology are related by the following exact sequence, which is a homology version of the homotopy sequences given in [13,14].

Proposition 3.11. *If X is an exterior space there is a long exact sequence*

$$\cdots \longrightarrow H_{q+1}^{seq}(X) \longrightarrow H_{q+1}^{tub}(X) \longrightarrow H_q^{seq}(X) \xrightarrow{(Sh)_*\text{-}id} H_q^{seq}(X) \longrightarrow \cdots.$$

Proof. Notice that the following short sequence is exact:

$$0 \longrightarrow S_\bullet^{seq}(X)^+ \xrightarrow{\alpha} S_{\bullet+1}^{tub}(X) \xrightarrow{\beta} S_\bullet^{seq}(X) \longrightarrow 0,$$

where $S_n^{seq}(X)^+ = S_{n+1}^{seq}(X)$ with $d_n^{seq+} = -d_{n+1}^{seq}(X)$, and $S_{\bullet+1}^{tub}(X)$ denotes the tubular complex with a dimension shift. On the other hand $\alpha(x) = (0, -x)$ and $\beta(a, x) = a$. Observe that the connecting homomorphism is $(Sh)_*$-id where $[x]$ leads to $[x(Sh) - x]$. □

3.3. Closed tubular homology

Let X be an exterior space. By its *closed tubular chain complex*, $S_\bullet^{ctu}(X)$, we mean the subcomplex of $S_\bullet^{tub}(X)$:

$$S_n^{ctu}(X) = \big\{(a, x) \in S_n^{tub}(X)\colon\ a_0 = 0\big\}.$$

If $f : X \to Y$ is an exterior map, $S_n^{ctu}(f)(a, x) = (S_{n-1}^{seq}(f)(a), S_n^{seq}(f)(x))$.

The nth *closed tubular homology* of X is the nth homology of the complex $S_\bullet^{ctu}(X)$, and it will be denoted by $H_n^{ctu}(X)$. We can extend this notion to exterior pairs without problems, giving rise to a new functor $H_n^{ctu}: \boldsymbol{E}^{(2)} \to \mathbf{Ab}$, defined for each integer n. One can check axioms (A1), (A2) and (A3). In order to compute the closed tubular homology of $\mathbb{N}$ we give a $\lim^1 - \lim$ relation for exterior spaces which are first countable at infinity.

Proposition 3.12. *Let X be an exterior space first countable at infinity and $X = E_0 \supset E_1 \supset E_2 \supset \cdots$ a fixed countable exterior base. Then there is a short exact sequence*:

$$0 \longrightarrow \lim{}^1\big\{H_{q+1}^{sin}(X, E_i)\big\} \xrightarrow{\alpha} H_q^{ctu}(X) \xrightarrow{\beta} \lim\big\{H_q^{sin}(X, E_i)\big\} \longrightarrow 0,$$

for all q.

Proof. We begin by defining β. Let (a, x) be a q-cycle of $S_\bullet^{ctu}(X)$. Then there is a monotone increasing sequence $\{n_i\}_{i=0}^{\infty} \subset \mathbb{N}$ with $n_0 = 0$, such that $(a_k, x_k) \in S_{q-1}^{sin}(E_i) \oplus S_q^{sin}(E_i)$ for all $k \geqslant n_i$. We consider $z_i = x_0 + x_1 + \cdots + x_{n_i-1} \in S_q^{sin}(X)$ and denote its equivalence class in $S_q^{sin}(X, E_i) = S_q^{sin}(X)/S_q^{sin}(E_i)$ by $\overline{z_i}$. Then we define $\beta([(a, x)]) = \{[\overline{z_i}]\}_{i=0}^{\infty} \in \lim\{H_q^{sin}(X, E_i)\}$.

On the other hand we also consider the homomorphism $\varphi: \prod_{i=0}^{\infty} H_{q+1}^{sin}(X, E_i) \to H_q^{ctu}(X)$ given as follows: if $\{[\overline{z_i}]\}_{i=0}^{\infty} \in \prod_{i=0}^{\infty} H_{q+1}^{sin}(X, E_i)$ then $x \in S_q^{seq}(X)$, where $x(e_i) = x_i = d_{q+1}(z_i) \in S_q^{sin}(E_i)$. Then $\varphi(\{[\overline{z_i}]\}_{i=0}^{\infty}) = [(0, x)]$. Since $\varphi d = 0$ there exists a unique homomorphism α such that $\alpha\pi = \varphi$.

We leave the details to the reader. □

Remark 3.13. There is a similar sequence for the relative case:

$$0 \longrightarrow \lim{}^1\big\{H_{q+1}^{sin}(X, E_i')\big\} \xrightarrow{\alpha} H_q^{ctu}(X, A) \xrightarrow{\beta} \lim\big\{H_q^{sin}(X, E_i')\big\} \longrightarrow 0,$$

where E_i' denotes $E_i \cap A$.

Corollary 3.14.

$$H_q^{ctu}(\mathbb{N}) = \begin{cases} \prod_0^{\infty} \mathbb{Z}, & \text{if } q = 0, \\ 0, & \text{otherwise.} \end{cases}$$

There exists a relationship between the tubular, closed tubular and singular homologies.

Proposition 3.15. *Let X be an exterior space. There is a long exact sequence*

$$\cdots \longrightarrow H_q^{ctu}(X) \longrightarrow H_q^{tub}(X) \longrightarrow H_{q-1}^{sin}(X) \longrightarrow H_{q-1}^{ctu}(X) \longrightarrow \cdots.$$

Proof. We only have to take into account that the following sequence of chain complexes is exact:

$$0 \longrightarrow S_\bullet^{ctu}(X) \xrightarrow{i} S_\bullet^{tub}(X) \xrightarrow{j} S_{\bullet-1}^{sin}(X) \longrightarrow 0,$$

where i is the canonical inclusion and j is defined by $j(a, x) = a_0$. □

Corollary 3.16. *If X is a topological space provided with the externology of all open subsets of X then $H_n^{ctu}(X) = H_n^{sin}(X)$.*

Hence, if X is a compact space, then $H_n^{ctu}(X_e) = H_n^{sin}(X)$. In particular, for the singleton space one has $H_n^{ctu}(P) = H_n^{sin}(P)$. The same holds for the relative case.

3.4. (Co)homology theories on $\boldsymbol{E}^{(2)}$ induced by (co)homology theories on $\mathbf{Top}^{(2)}$

Another example of (co)homology theory is the following. For a given (co)homology theory on $\mathbf{Top}^{(2)}$ in the sense of Eilenberg–Steenrod [5], we have a (co-)homology theory on $\boldsymbol{E}^{(2)}$ taking into account that there exists a forgetful functor $\boldsymbol{E}^{(2)} \to \mathbf{Top}^{(2)}$.

3.5. Sequential (co)homology with coefficients in an $\mathcal{R}$-module

Definition 3.17. Let $\mathcal{A}$ be an Abelian category with sufficient projectives. We consider the category of bounded below chain complexes on $\mathcal{A}$, $\mathbf{Ch_{bb}}(\mathcal{A})$. If $f : X \to Y$ is a chain map in $\mathbf{Ch_{bb}}(\mathcal{A})$, then f is said to be a

- *fibration*, if each f_n is an epimorphism,
- *cofibration*, if each f_n is a monomorphism and $\operatorname{coker}(f_n)$ is projective; and
- *weak equivalence*, if f induces isomorphisms in homology.

It is well known that $\mathbf{Ch_{bb}}(\mathcal{A})$, with the fibrations, cofibrations and weak equivalences given above, has a closed model category structure in the sense of Quillen [15].

Then, if (X, A) is an exterior pair, for the natural inclusion $i : S_\bullet^{seq}(A) \to S_\bullet^{seq}(X)$ we can construct a commutative square of the form:

$$\begin{array}{ccc} {}^{cof}S_\bullet^{seq}(A) & \xrightarrow{{}^{cof}i} & {}^{cof}S_\bullet^{seq}(X) \\ \downarrow{\scriptstyle p_A} & & \downarrow{\scriptstyle p_X} \\ S_\bullet^{seq}(A) & \xrightarrow[i]{} & S_\bullet^{seq}(X) \end{array}$$

where ${}^{cof}S_\bullet^{seq}(A)$, ${}^{cof}S_\bullet^{seq}(X)$ are cofibrant chain complexes, p_A, p_X are trivial fibrations and ${}^{cof}i$ is a cofibration. We consider ${}^{cof}S_\bullet^{seq}(X, A)$ the coker of ${}^{cof}i$.

Definition 3.18. Let $\mathfrak{M}$ be a left $\mathcal{R}$-module. The chain complex

$$S_\bullet^{seq}(X, A; \mathfrak{M}) = {}^{cof}S_\bullet^{seq}(X, A) \otimes_{\mathcal{R}} \mathfrak{M}$$

is said to be the *sequential chain complex with coefficients* in $\mathfrak{M}$ of (X, A).

It is easy to check that this construction gives, for each exterior map $f : (X, A) \to (Y, B)$, a chain map $S_\bullet^{seq}(f; \mathfrak{M}) : S_\bullet^{seq}(X, A; \mathfrak{M}) \to S_\bullet^{seq}(Y, B; \mathfrak{M})$ up to homotopy and, using a sufficiently strong axiom of choice, a functor

$$H_n^{seq}(.; \mathfrak{M}) : \boldsymbol{E}^{(2)} \to \mathcal{R}\text{-}\mathbf{Mod},$$

as well as a natural transformation $\partial : H_n^{seq}(X, A; \mathfrak{M}) \to H_{n-1}^{seq}(A; \mathfrak{M})$.

$H_n^{seq}(X, A; \mathfrak{M})$ will be called the nth *sequential homology with coefficients* in $\mathfrak{M}$ of (X, A).

Taking into account that $S_\bullet^{seq}(X, A)$ is a pseudoprojective chain complex (that is, for all n, $S_n^{seq}(X, A)$ is a projective $\mathcal{R}$-module) it is not difficult to prove (and is left as an exercise) the following result:

Theorem 3.19. *Sequential homology with coefficients in $\mathfrak{M}$ is a homology theory on $\boldsymbol{E}^{(2)}$ with $H_0^{seq}(\mathbb{N}; \mathfrak{M}) = \mathfrak{M}$.*

Remark 3.20. Observe that, in a natural way, we can define sequential cohomology with coefficients in a left $\mathcal{R}$-module $\mathfrak{M}$ using the cochain complex $\mathrm{Hom}_{\mathcal{R}\text{-}\mathbf{Mod}}({}^{cof}S_\bullet^{seq}(X, A), \mathfrak{M})$, giving rise to a cohomology theory on $\boldsymbol{E}^{(2)}$.

4. Applications

4.1. Closed tubular homology and locally finite homology

To compute the cellular closed tubular homology we will use the following:

Proposition 4.1. *Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a countable family of compact spaces and consider on $\coprod_{\lambda\in\Lambda} X_\lambda$ as e-open subsets the complements of all closed-compact subsets. Then*

$$H_n^{ctu}\left(\coprod_{\lambda\in\Lambda} X_\lambda\right) \cong \prod_{\lambda\in\Lambda} H_n^{sin}(X_\lambda).$$

Proof. The case in which Λ is finite is straightforward from the properties of a homology theory on $\boldsymbol{E}^{(2)}$.

Suppose that $\Lambda = \mathbb{N}$. We observe that

$$\coprod_{\lambda=0}^{\infty} X_\lambda \supset \coprod_{\lambda=1}^{\infty} X_\lambda \supset \coprod_{\lambda=2}^{\infty} X_\lambda \supset \cdots$$

is an infinite countable exterior base for $\coprod_{\lambda=0}^{\infty} X_\lambda$. Applying Proposition 3.12 one has the desired result. □

Let $\bar{e}_\lambda^n$, $\dot{e}_\lambda^n$ denote an n-cell and its boundary, and $e_\lambda^n = \bar{e}_\lambda^n - \dot{e}_\lambda^n$. Suppose X is oriented, that is, we have chosen a generator a_λ^n for each infinite cyclic group $H_n^{sin}(D_\lambda^n, S_\lambda^{n-1}) \cong H_n^{sin}(\bar{e}_\lambda^n, \dot{e}_\lambda^n)$. We recalled in Section 1.2.2 that the locally finite homology of a strongly locally finite CW-complex X is given by the chain complex $C_\bullet^{lf}(X)$, defined as:

$$C_n^{lf}(X) = \prod_{\lambda\in A_n} H_n^{sin}\big(D_\lambda^n, S_\lambda^{n-1}\big) \cong \prod_{\lambda\in A_n} \mathbb{Z}.$$

Using the excision axiom and the proposition above, for the associated exterior space $\overline{X}$, one has a commutative diagram

$$\begin{array}{ccc} H_n^{ctu}(\overline{X}_n, \overline{X}_{n-1}) & \xrightarrow{\cong} & \prod_{\lambda \in A_n} H_n^{sin}(D_\lambda^n, S_\lambda^{n-1}) \\ \downarrow d_n^{ctu} & & \downarrow -d_n^{lf} \\ H_n^{ctu}(\overline{X}_{n-1}, \overline{X}_{n-2}) & \xrightarrow[\cong]{} & \prod_{\mu \in A_{n-1}} H_{n-1}^{sin}(D_\mu^{n-1}, S_\mu^{n-2}) \end{array}$$

Then one obtains the following:

Theorem 4.2. *If X is a strongly locally finite CW-complex, then for each integer n its locally finite homology is isomorphic to its closed tubular homology*:

$$H_n^{lf}(X) \cong H_n^{ctu}(\overline{X}).$$

4.2. Tubular homology and end homology

The end homology (cf. Section 1.2.3) of a strongly locally finite CW-complex X is given by the chain complex $C_\bullet^\infty(X)$, defined as:

$$C_n^\infty(X) = \prod_{\lambda \in A_n} H_n^{sin}(D_\lambda^n, S_\lambda^{n-1}) / \bigoplus_{\lambda \in A_n} H_n^{sin}(D_\lambda^n, S_\lambda^{n-1}),$$

and from the commutative diagram

$$\begin{array}{ccc} H_n^{tub}(\overline{X}_n, \overline{X}_{n-1}) & \xrightarrow{\cong} & \prod_{\lambda \in A_n} H_n^{sin}(D_\lambda^n, S_\lambda^{n-1}) / \bigoplus_{\lambda \in A_n} H_n^{sin}(D_\lambda^n, S_\lambda^{n-1}) \\ \downarrow d_n^{tub} & & \downarrow -d_n^\infty \\ H_n^{tub}(\overline{X}_{n-1}, \overline{X}_{n-2}) & \xrightarrow[\cong]{} & \prod_{\mu \in A_{n-1}} H_{n-1}^{sin}(D_\mu^{n-1}, S_\mu^{n-2}) / \bigoplus_{\mu \in A_{n-1}} H_{n-1}^{sin}(D_\mu^{n-1}, S_\mu^{n-2}) \end{array}$$

the following follows.

Theorem 4.3. *If X is a strongly locally finite CW-complex, then for each integer n its end homology is isomorphic to its tubular homology*:

$$H_n^\infty(X) \cong H_n^{tub}(\overline{X}).$$

4.3. Comparison of sequential homology with singular, locally finite and end homology

We note that the Abelian groups $\bigoplus_0^\infty \mathbb{Z}$, $\prod_0^\infty \mathbb{Z}$ and $\prod_0^\infty \mathbb{Z} / \bigoplus_0^\infty \mathbb{Z}$ have the structure of left $\mathcal{R}$-modules given as follows: An element of $\mathcal{R}$ is given by a locally finite $(\infty \times \infty)$-matrix and an element of $\bigoplus_0^\infty \mathbb{Z}$ can be represented by a $(\infty \times 1)$-matrix. The action of the ring on the Abelian group is induced by matrix multiplication, and similarly for the other Abelian groups.

Given a strongly locally finite CW-complex X, and the associated exterior space $\overline{X}$, one has natural isomorphisms

$$H_n^{seq}(\overline{X}_n, \overline{X}_{n-1}) \otimes_{\mathcal{R}} \left(\bigoplus_0^\infty \mathbb{Z}\right) \cong H_n^{sin}(X_n, X_{n-1}),$$

$$H_n^{seq}\big(\overline{X}_n, \overline{X}_{n-1}\big) \otimes_{\mathcal{R}} \left(\prod_0^\infty \mathbb{Z}\right) \cong H_n^{lf}(X_n, X_{n-1}),$$
$$H_n^{seq}\big(\overline{X}_n, \overline{X}_{n-1}\big) \otimes_{\mathcal{R}} \left(\prod_0^\infty \mathbb{Z}/\bigoplus_0^\infty \mathbb{Z}\right) \cong H_n^{\infty}(X_n, X_{n-1}),$$

that commute with boundary operators. Then one has:

Theorem 4.4. *Let X be a strongly locally finite CW-complex. Then for each integer n, one has the following isomorphisms*:

(i) $H_n^{seq}(\overline{X}; \bigoplus_0^\infty \mathbb{Z}) \cong H_n^{sin}(X)$,
(ii) $H_n^{seq}(\overline{X}; \prod_0^\infty \mathbb{Z}) \cong H_n^{lf}(X)$,
(iii) $H_n^{seq}(\overline{X}; \prod_0^\infty \mathbb{Z}/\bigoplus_0^\infty \mathbb{Z}) \cong H_n^{\infty}(X)$.

Now, using the results of the subsection above, one has:

Corollary 4.5. *Let X be a strongly locally finite CW-complex. Then for each integer n, one has the following isomorphisms*:

(i) $H_n^{seq}(\overline{X}; \prod_0^\infty \mathbb{Z}) \cong H_n^{ctu}(\overline{X})$,
(ii) $H_n^{seq}(\overline{X}; \prod_0^\infty \mathbb{Z}/\bigoplus_0^\infty \mathbb{Z}) \cong H_n^{tub}(\overline{X})$.

4.4. Comparison of sequential cohomology with compact support, singular and end cohomology

The Abelian groups $\bigoplus_0^\infty \mathbb{Z}$, $\prod_0^\infty \mathbb{Z}$ and $\prod_0^\infty \mathbb{Z}/\bigoplus_0^\infty \mathbb{Z}$ also have the structure of right $\mathcal{R}$-module induced by matrix multiplication.

Given a strongly locally finite CW-complex X, one has natural isomorphisms

$$\mathrm{Hom}_{\mathcal{R}}\left(H_n^{seq}\big(\overline{X}_n, \overline{X}_{n-1}\big), \bigoplus_0^\infty \mathbb{Z}\right) \cong H_{cs}^n(X_n, X_{n-1}),$$
$$\mathrm{Hom}_{\mathcal{R}}\left(H_n^{seq}\big(\overline{X}_n, \overline{X}_{n-1}\big), \prod_0^\infty \mathbb{Z}\right) \cong H_{sing}^n(X_n, X_{n-1}),$$
$$\mathrm{Hom}_{\mathcal{R}}\left(H_n^{seq}\big(\overline{X}_n, \overline{X}_{n-1}\big), \prod_0^\infty \mathbb{Z}/\bigoplus_0^\infty \mathbb{Z}\right) \cong H_n^{\infty}(X_n, X_{n-1}),$$

that commute with boundary operators. Then one has:

Theorem 4.6. *Let X be a strongly locally finite CW-complex. Then for each integer n, one has the following isomorphisms*:

(i) $H^n_{seq}(\overline{X}; \bigoplus_0^\infty \mathbb{Z}) \cong H^n_{cs}(X)$,
(ii) $H^n_{seq}(\overline{X}; \prod_0^\infty \mathbb{Z}) \cong H^n_{sing}(X)$,
(iii) $H^n_{seq}(\overline{X}; \prod_0^\infty \mathbb{Z}/\bigoplus_0^\infty \mathbb{Z}) \cong H^n_{\infty}(X)$.

4.5. Comparison of sequential homology with Čech and Steenrod homology

Given a compact metric space X, one can consider the open fundamental complex $OFC(X)$ introduced by Lefschetz [10], which is a strongly locally finite complex. The sequential homology and Čech homology are related as follows: We can consider the shift operator $Sh : \bigoplus_0^\infty \mathbb{Z} \to \bigoplus_0^\infty \mathbb{Z}$ given by $Sh(z_0, z_1, z_2, \ldots) = (z_1, z_2, \ldots)$ and $id\text{-}Sh$ which is an element of the ring $\mathcal{R}$. We take into account the fact that the homology group $H^n_{seq}(X)$ has the structure of an $\mathcal{R}$-module. It is easy to check the following:

Theorem 4.7. *Let X be a compact metric space and consider the exterior space $\overline{OFC(X)}$. Then*

$$\check{H}_n(X) \cong \bigl\{x \in H_n^{seq}(\overline{OFC(X)}) / x(id\text{-}Sh) = 0\bigr\}.$$

With respect to Steenrod homology one has the following:

Theorem 4.8. *Let X be a compact metric space and consider the exterior space $\overline{OFC(X)}$. Then*

(i) *for any integer n, $H^{seq}_{n+1}(\overline{OFC(X)}; \prod_0^\infty \mathbb{Z}) \cong H^{ctu}_{n+1}(\overline{OFC(X)}) \cong \widetilde{H}^{St}_n(X)$,*
(ii) *for any integer n, $H^{seq}_{n+1}(\overline{OFC(X)}; \prod_0^\infty \mathbb{Z}/\bigoplus_0^\infty \mathbb{Z}) \cong H^{tub}_{n+1}(\overline{OFC(X)}) \cong H^{St}_n(X)$.*

Remark 4.9. Notice that $H_0^{ctu}(\overline{OFC(X)}) \cong H_0^{tub}(\overline{OFC(X)}) \cong 0$.

4.6. Comparison of sequential cohomology with Čech cohomology

Using the notation of the subsection above, one has:

Theorem 4.10. *Let X be a compact metric space and consider the exterior space $\overline{OFC(X)}$. Then*

(i) *for any integer n, $H^{n+1}_{seq}(\overline{OFC(X)}; \bigoplus_0^\infty \mathbb{Z}) \cong \widetilde{\check{H}}{}^n(X)$,*
(ii) *for any integer n, $H^n_{seq}(\overline{OFC(X)}; \prod_0^\infty \mathbb{Z}/\bigoplus_0^\infty \mathbb{Z}) \cong \check{H}^n(X)$.*

4.7. Poincaré duality

In the ring $\mathcal{R}$ of locally finite matrices one has the transposition antihomomorphism, which applies a matrix r to its transpose and is denoted by r^t. A right $\mathcal{R}$-module also admits the structure of a left $\mathcal{R}$-module by the action $r \cdot m = m \cdot r^t$.

Theorem 4.11.
Let M be a triangulable, second countable, orientable, n-manifold. Then

(i) $H^q_{seq}(\overline{M}) \cong H^{seq}_{n-q}(\overline{M})$;
(ii) *for any right $\mathcal{R}$-module $\mathfrak{M}$, $H^q_{seq}(\overline{M}; \mathfrak{M}) \cong H^{seq}_{n-q}(\overline{M}; \mathfrak{M})$, where in the second part of the isomorphism $\mathfrak{M}$ is considered as a left $\mathcal{R}$-module.*

Proof. Denote by M' the dual triangulation of the n-manifold M. Taking into account the properties of incidence of dual cells one has that the following diagram,

$$\begin{array}{ccc} H^r_{seq}(\overline{M}_r, \overline{M}_{r-1}; \mathfrak{M}) & \longrightarrow & H^{seq}_{n-r}(\overline{M'}_{n-r}, \overline{M'}_{n-r-1}; \mathfrak{M}) \\ \downarrow & & \downarrow \\ H^{r+1}_{seq}(\overline{M}_{r+1}, \overline{M}_r; \mathfrak{M}) & \longrightarrow & H^{seq}_{n-r-1}(\overline{M'}_{n-r-1}, \overline{M'}_{n-r-2}; \mathfrak{M}) \end{array}$$

which commutes perhaps up to a sign. Therefore one obtains (ii), and for $\mathfrak{M} = \mathcal{R}$, (i) follows. □

Corollary 4.12. *Let M be a triangulable, second countable, orientable, n-manifold. Then*

(i) $H^q_{cs}(M) \cong H^q_{seq}(\overline{M}; \bigoplus_0^\infty \mathbb{Z}) \cong H^{seq}_{n-q}(\overline{M}; \bigoplus_0^\infty \mathbb{Z}) \cong H^{sin}_{n-q}(M)$,

(ii) $H^q_{sin}(M) \cong H^q_{seq}(\overline{M}; \prod_0^\infty \mathbb{Z}) \cong H^{seq}_{n-q}(\overline{M}; \prod_0^\infty \mathbb{Z}) \cong H^{lf}_{n-q}(M)$,

(iii) $H^q_\infty(M) \cong H^q_{seq}(\overline{M}; \prod_0^\infty \mathbb{Z}/\bigoplus_0^\infty \mathbb{Z}) \cong H^{seq}_{n-q}(\overline{M}; \prod_0^\infty \mathbb{Z}/\bigoplus_0^\infty \mathbb{Z}) \cong H^\infty_{n-q}(M)$.

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