Article

Augmented Simplicial Combinatorics through Category Theory: Cones, Suspensions and Joins

José Manuel García-Calcines [1,*,†,‡] , **Luis Javier Hernández-Paricio** [2,‡,§] and **María Teresa Rivas-Rodríguez** [2,‡,§]

1 Department de Matemáticas, Estadística e I.O., Facultad de Ciencias (Sección de Matemáticas), Campus de Anchieta, University of La Laguna, 38200 La Laguna, Spain
2 Department de Matemáticas y Computación, Facultad de Ciencia y Tecnología, University of La Rioja, 26006 Logroño, Spain; luis-javier.hernandez@unirioja.es (L.J.H.-P.); maria-teresa.rivas@unirioja.es (M.T.R.-R.)
* Correspondence: jmgarcal@ull.edu.es; Tel.: +34-922-318-150
† Current address: Avda. Astrofísico Fco. Sánchez, s/n., Aptado. 456, 38200 La Laguna, Spain.
‡ These authors contributed equally to this work.
§ Current address: C/Madre de Dios 53, Edificio CCT, 26006 Logroño, Spain.

Abstract: In this work, we analyze the combinatorial properties of the category of augmented semi-simplicial sets. We consider various monoidal structures induced by the co-product, the product, and the join operator in this category. In addition, we also consider monoidal structures on augmented sequences of integers induced by the sum and product of integers and by the join of augmented sequences. The cardinal functor that associates to each finite set X its cardinal $|X|$ induces the sequential cardinal that associates to each augmented semi-simplicial finite set X an augmented sequence $|X|_n$ of non-negative integers. We prove that the sequential cardinal functor is monoidal for the corresponding monoidal structures. This allows us to easily calculate the number of simplices of cones and suspensions of an augmented semi-simplicial set as well as other augmented semi-simplicial sets which are built by joins. In this way, the monoidal structures of the augmented sequences of numbers may be thought of as an algebraization of the augmented semi-simplicial sets that allows us to do a simpler study of the combinatorics of the augmented semi-simplicial finite sets.

Keywords: augmented semi-simplicial set; augmented integer sequence; monoidal category; simplicial combinatorics

1. Introduction

The main objectives of this work are framed in the study of the combinatorics of augmented semi-simplicial sets. We can highlight, among others, the following targets:

- The study of some properties about the combinatorics of the faces in semi-simplicial complexes (polyhedra)
- The analysis of the numerical sequences (sequential cardinals) arising from the combinatorial structure of semi-simplicial complexes.
- The development of some semi-simplicial constructions and the search for methods for counting the number of their faces.
- The study of relationships between such semi-simplicial constructions and their associated numerical sequences.

We are considering two kind of mathematical objects:

(i) Augmented semi-simplicial and co-semi-simplicial objects,
(ii) Augmented integer sequences and matrices.

As far as simplicial objects are concerned, we have focused on what we have called Γ_+^{op}-sets, or augmented semi-simplicial sets. Using this category has its advantages and drawbacks. On the one hand, we point out that the significant information is given in

the combinatorics of non-degenerated simplices so that fewer data are necessary for their coding. Nevertheless, the use of degenerated simplices would allow one to represent simplicially a greater number of continuous maps. Another feature in our study is considering augmented semi-simplicial objects. This slight modification makes both the structures and formulas we are using symmetric and much more reduced, so that it helps facilitate their computation. For notions related with semi-simplicial sets, we refer the reader to [1–5] and for a study of the realization of semi-simplicial sets [6]. For category models related with homotopy theory of simplicial sets and topological spaces, see [7].

As basic combinatorial elements, we are considering the following ones:

(bc) Binomial coefficients (If $q > p$, take $\binom{p}{q} = 0$):

$$\binom{p}{q} = \frac{p!}{q!(p-q)!} = \frac{p(p-1)\cdots(p-q+1)}{(p-q)!}$$

which give the number of strictly increasing maps from the ordered set with q elements $\{1 < \cdots < q\}$ to the ordered set of p elements $\{1 < \cdots < p\}$. These numbers occur as coefficients in Newton's binomial formula

$$(a+b)^p = \sum_{q=0}^{p} \binom{p}{q} a^{p-q} b^q$$

and the coefficients in Pascal's triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 2 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots
\end{array}
$$

which can also represented by the matrix

$$
\begin{pmatrix}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots & \\
\hline
0| & \binom{0}{0} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
1| & \binom{1}{0} & \binom{1}{1} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
2| & \binom{2}{0} & \binom{2}{1} & \binom{2}{2} & 0 & 0 & 0 & 0 & \cdots & 0 \\
3| & \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} & 0 & 0 & 0 & \cdots & 0 \\
4| & \binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} & \binom{4}{4} & 0 & 0 & \cdots & 0 \\
5| & \binom{5}{0} & \binom{5}{1} & \binom{5}{2} & \binom{5}{3} & \binom{5}{4} & \binom{5}{5} & 0 & \cdots & 0 \\
6| & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}
$$

In order to write this work, we have examined some results about binomial numbers in the following references: [8–12].

(gc) Numbers associated with certain geometric configurations: Consider integers $d \geq 0$ and $n \geq 1$. The *d-dimensional n-th triangular number* T_n^d is inductively defined as follows:

$$
\begin{array}{ll}
T_n^0 = 1, & \text{if } d = 0 \\
T_n^d = \sum_{i=1}^{n} T_i^{d-1}, & \text{if } d \geq 1.
\end{array}
$$

These numbers are contained in the family of *regular polytope* numbers , see [13].

When you consider other combinatorial subjects on semi-simplicial sets, there are many connections with the standard families of sequences and matrices of numbers as Stirling [14], and chain-power-set numbers [15]. For more information about these types of

numbers, we refer the reader to [16,17]. Nevertheless, in order to write a shorter paper, we have focused on analyzing some relations between semi-simplicial finite sets and binomial and triangular numbers.

In this work, we study a new version of join construction for augmented semi-simplicial sets which has similar properties to the one developed by P. Ehlers and T. Porter [18] for augmented simplicial sets. However, one important novelty of our work is that we endow the category of augmented semi-simplicial finite sets with a categorical semi-ring structure (i.e., a bimonoidal category structure) that admits a right action of augmented co-semi-simplicial objects in the category of semi-simplicial finite sets. This structural enrichment permits an interesting study of the combinatorial properties of the join operations and constructions obtained by actions. The categorical semi-rings used in this paper are also called bimonoidal categories [19]. Recent results on these categorical structures can be seen in [20,21]. Our paper gives interesting new applications of these structures to the study of combinatorial problems of augmented semi-simplicial finite sets.

In the algebraic context, we consider categories of augmented sequences of integers with a categorical ring structure that admits actions of certain augmented matrices. This action is defined by using the inverse matrix of the matrix associated with binomial transformations.

The sequential cardinal functor, denoted by $|\cdot|$, applies a semi-simplicial finite set X to the sequence $|X|_n := |X_n|$, where X_n denotes the set of n-simplices of X and $|X_n|$ is its cardinal. The main results of this paper ensure, on the one hand, that the sequential cardinal functor is a homomorphism of categorical semi-rings (Theorem 4) and, on the other hand, that the sequential cardinal functor is compatible with action operators (Theorem 5, Corollary 6). These results allows us to easily count the number of simplices of a semi-simplicial finite set built through join and action operations.

A (augmented) simplicial complex is a family of subsets of a finite set F, $\Sigma \subset \mathcal{P}(F)$, such that, if $\sigma \in \Sigma$ and $\varnothing \neq \sigma' \subset \sigma$ ($\sigma' \subset \sigma$), then $\sigma' \in \Sigma$. We note that a (augmented) simplicial complex Σ has a canonical structure of (augmented) semi-simplicial set Σ', where $\Sigma'_k = \{\sigma \in \Sigma | |\sigma| = k+1\}$, $k \geq 0$ ($k \geq -1$). The f-vector of a (augmented) simplicial complex $f = f(\Sigma) = (f_0, \cdots f_n)$ ($f = f(\Sigma) = (f_{-1}, f_0, \cdots f_n)$) of a (augmented) simplicial complex is given by $f_k = |\Sigma'|_k$. Therefore, the notion of f-vector is obtained as a restriction of sequential cardinal (given in the present work) to augmented simplicial complexes. For a study of the main properties of f-vectors and their associated h-vectors, we refer the reader to [22] and for recent advances on the study of f-vectors: [23–25]. In our work, instead of studying realization problems, we prove new results for semi-simplicial finite sets by taking the sequential cardinal and obtaining new properties related to join and action operations on augmented semi-simplicial finite sets.

At the end of the work, we give a concrete example of calculations associated with constructions given by joins and actions induced by the dunce cap. These calculations are done by using the inverse matrix of augmented binomial numbers and the augmented matrix of triangular numbers (Corollaries 12 and 13).

2. Categorical Preliminaries: Presheaves and Monoidal Categories
2.1. Extension of a Small Category to a Cocomplete One Using Presheaves

We will denote as **Sets** the category of sets.

Given a small category **C**, we can consider its opposite category $\mathbf{C}^{op}$. Then, the usual functor category $\mathbf{Sets}^{\mathbf{C}^{op}}$ has as objects functors $X : \mathbf{C}^{op} \to \mathbf{Sets}$ and as arrows $X \to X'$ all natural transformations $f : X \to X'$ between such functors. The category $\mathbf{Sets}^{\mathbf{C}^{op}}$ is usually called *category of presheaves on* **C**.

For a given object c in **C**, we can consider $\mathbf{Y}(c)$, the presheaf on **C** defined as the contravariant Hom-functor $\mathbf{Y}(c)(-) := \mathrm{Hom}_\mathbf{C}(-, c)$. This construction gives rise to the well-known Yoneda embedding

$$\mathbf{Y} : \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{op}}, \quad c \mapsto \mathrm{Hom}_\mathbf{C}(-, c).$$

By an embedding in this setting, we mean a full and faithful functor. Any presheaf which is isomorphic to a presheaf of the form $\mathbf{Y}(c)$ is called *representable*. The *Yoneda lemma* is well known, asserting that, for any presheaf, X, there exists a bijection between the natural transformations $\mathbf{Y}(c) \to X$ and the elements of $X(c)$:

$$\mathbf{Nat}(\mathbf{Y}(c), X) \overset{\cong}{\to} X(c), \quad \alpha \mapsto \alpha_c(1_c).$$

Associated with $X \in \mathbf{Sets}^{\mathbf{C}^{op}}$, we have the so-called *category of elements* of X, denoted by $\int_{\mathbf{C}} X$. Its objects are all pairs (c, x), where c is an object in $\mathbf{C}$ and $x \in X(c)$; a morphism $(c, x) \to (c', x')$ in $\int_{\mathbf{C}} X$ consists of a morphism $\varphi \colon c \to c'$ in $\mathbf{C}$ such that $X(\varphi)(x') = x$. Observe that we have a projection functor

$$\pi_X \colon \int_{\mathbf{C}} X \to \mathbf{C}, \quad (c, x) \mapsto c.$$

The proof of the following theorem can be found in [26].

Theorem 1. *Let $Z \colon \mathbf{C} \to \mathcal{E}$ be a functor from a small category $\mathbf{C}$ to a cocomplete category $\mathcal{E}$. Then, the functor $Sing^Z \colon \mathcal{E} \to \mathbf{Sets}^{\mathbf{C}^{op}}$ given by*

$$Sing^Z(Y)(c) := Hom_{\mathcal{E}}(Z(c), Y)$$

admits a left adjoint functor $L^Z \colon \mathbf{Sets}^{\mathbf{C}^{op}} \to \mathcal{E}$, defined for each presheaf X as

$$L^Z(X) := colim(\int_{\mathbf{C}} X \xrightarrow{\pi_X} \mathbf{C} \xrightarrow{Z} \mathcal{E})$$

The functor L^Z preserves colimits and makes commutative the following diagram

In other words, L^Z is an extension of Z that preserves colimits. Moreover L^Z is, up to natural isomorphism, the only extension of Z preserving all colimits.

Remark 1. *Observe that, actually, we have a functor:*

$$\mathbf{Sets}^{\mathbf{C}^{op}} \times \mathcal{E}^{\mathbf{C}} \to \mathcal{E}, \quad (X, Z) \mapsto L^Z(X).$$

As a particular case, taking $\mathcal{E} = \mathbf{Sets}^{\mathbf{C}^{op}}$ and $Z = \mathbf{Y}$ the Yoneda embedding, we have that the functor $Sing^Z \colon \mathbf{Sets}^{\mathbf{C}^{op}} \to \mathbf{Sets}^{\mathbf{C}^{op}}$ is naturally isomorphic to the identity functor (by the Yoneda lemma). Therefore, L^Z must be also naturally isomorphic to the identity functor so we obtain the following corollary:

Corollary 1. *Every presheaf X on $\mathbf{C}$ is, up to natural isomorphism, a colimit of representable presheaves:*

$$X \cong colim(\int_{\mathbf{C}} X \xrightarrow{\pi_X} \mathbf{C} \xrightarrow{\mathbf{Y}} \mathbf{Sets}^{\mathbf{C}^{op}}).$$

We also observe that, as a consequence of Theorem 1 above with $\mathcal{E} = \mathbf{Sets}^{\mathbf{C}^{op}}$, one has certain suitable induced functors. Such functors will receive the name of action functors:

Definition 1. *The action functors are the following:*

1. $(-)\widetilde{\rhd}(-)\colon \mathbf{Sets}^{\mathbf{C}^{op}} \times (\mathbf{Sets}^{\mathbf{C}^{op}})^{\mathbf{C}} \to \mathbf{Sets}^{\mathbf{C}^{op}}$

$$X\widetilde{\rhd}Y := L^Y(X).$$

2. $(-)\widetilde{\rhd}(-)\colon (\mathbf{Sets}^{\mathbf{C}^{op}})^{\mathbf{C}} \times (\mathbf{Sets}^{\mathbf{C}^{op}})^{\mathbf{C}} \to (\mathbf{Sets}^{\mathbf{C}^{op}})^{\mathbf{C}}$

$$(Y\widetilde{\rhd}Z)(c) := Y(c)\widetilde{\rhd}Z = L^Z(Y(c)).$$

Given $X \in \mathbf{Sets}^{\mathbf{C}^{op}}$ and $Y, Z \in (\mathbf{Sets}^{\mathbf{C}^{op}})^{\mathbf{C}}$, the object $X\widetilde{\rhd}Y \in \mathbf{Sets}^{\mathbf{C}^{op}}$ is said to be the *(right) action of Y on X*. Similarly, $Y\widetilde{\rhd}Z \in (\mathbf{Sets}^{\mathbf{C}^{op}})^{\mathbf{C}}$ is said to be the *(right) action of Z on Y*.

Remark 2. *If $X \in \mathbf{Sets}^{\mathbf{C}^{op}}$ and $Y, Z \in (\mathbf{Sets}^{\mathbf{C}^{op}})^{\mathbf{C}}$, then it follows that*

$$(X\widetilde{\rhd}Y)\widetilde{\rhd}Z \cong X\widetilde{\rhd}(Y\widetilde{\rhd}Z).$$

This formula inspires the name given: "action functors".

2.2. Monoidal Categories

In this subsection, we include the definition of monoidal category and that of functors between monoidal categories. For a more complete study on the notions and properties related to monoidal categories, we recommend reading the volumes by Niles Johnson and Donald Yau [19].

Definition 2. *A monoidal category is a category $\mathbf{C}$ equipped with:*

(1) *a functor $\otimes\colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ out of the product category of $\mathbf{C}$ with itself, called the tensor product,*
(2) *an object I called the unit object or tensor unit, and*
(3) *three natural isomorphisms:*
     The associator α, with components

$$\alpha_{A,B,C}\colon (A \otimes B) \otimes C \xrightarrow{\cong} A \otimes (B \otimes C);$$

The left unitor λ and the right unitor ρ, with components

$$\lambda_A\colon I \otimes A \xrightarrow{\cong} A, \quad \rho_A\colon A \otimes I \xrightarrow{\cong} A,$$

such that the following diagrams are commutative (coherence diagrams):
Pentagon identity:

Triangle identity:

By a strict monoidal category, we mean a monoidal category in which the natural isomorphisms α, λ, ρ are identities. In this case, the pentagon and triangle diagrams commute automatically.

Definition 3. *A symmetric monoidal category is a monoidal category $(\mathbf{C}, \otimes, I)$ having a natural transformation s (it is necessarily a natural isomorphism) called the braiding, with components $s_{A,B} \colon A \otimes B \to B \otimes A$, making commutative the following kind of diagram:*

$$
\begin{array}{ccc}
 & B \otimes A & \\
{\scriptstyle s_{A,B}}\nearrow & & \searrow{\scriptstyle s_{B,A}} \\
A \otimes B & \xrightarrow[\text{id}]{} & A \otimes B
\end{array}
$$

That is, $s_{A,B}^{-1} = s_{B,A}$. Moreover, we also demand that the braiding and the associator obey the hexagon identity:

$$
\begin{array}{ccc}
(A \otimes B) \otimes C \xrightarrow{\alpha} A \otimes (B \otimes C) \xrightarrow{s} (B \otimes C) \otimes A \\
{\scriptstyle s \otimes 1}\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle \alpha} \\
(B \otimes A) \otimes C \xrightarrow{\alpha} B \otimes (A \otimes C) \xrightarrow{1 \otimes s} B \otimes (C \otimes A)
\end{array}
$$

If, in addition, the monoidal category is strict, we will say that $\mathbf{C}$ is a strict symmetric monoidal category.

Definition 4. *A functor $F \colon \mathbf{C} \to \mathbf{C}'$ between monoidal categories is said to be 2-monoidal if it is equipped with a natural isomorphism Φ with components $\Phi_{A,B} \colon F(A) \otimes F(B) \xrightarrow{\cong} F(A \otimes B)$, such that the following diagram commutes for any objects A, B, C in $\mathbf{C}$:*

$$
\begin{array}{ccc}
(F(A) \otimes F(B)) \otimes F(C) \xrightarrow{\Phi \otimes 1} F(A \otimes B) \otimes F(C) \xrightarrow{\Phi} F((A \otimes B) \otimes C) \\
{\scriptstyle \alpha}\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle F(\alpha)} \\
F(A) \otimes (F(B) \otimes F(C)) \xrightarrow[1 \otimes \Phi]{} F(A) \otimes F(B \otimes C) \xrightarrow[\Phi]{} F(A \otimes (B \otimes C))
\end{array}
$$

We say that F is $(2,0)$-monoidal (or just monoidal in classical terminology) if, in addition, there exists an isomorphism $\phi \colon I \xrightarrow{\cong} F(I)$ satisfying that the following diagrams are commutative for any object A in $\mathbf{C}$:

$$
\begin{array}{ccc}
I \otimes F(A) \xrightarrow{\lambda} F(A) & \qquad & F(A) \otimes I \xrightarrow{\rho} F(A) \\
{\scriptstyle \phi \otimes 1}\downarrow \qquad\qquad \uparrow{\scriptstyle F(\lambda)} & & {\scriptstyle 1 \otimes \phi}\downarrow \qquad\qquad \uparrow{\scriptstyle F(\rho)} \\
F(I) \otimes F(A) \xrightarrow[\Phi]{} F(I \otimes A) & & F(A) \otimes F(I) \xrightarrow[\Phi]{} F(A \otimes I)
\end{array}
$$

Finally, a 2-monoidal (or $(2,0)$-monoidal) functor $F \colon \mathbf{C} \to \mathbf{C}'$ between symmetric monoidal categories is said to be symmetric if it satisfies the following commutative diagram, for any objects A, B in $\mathbf{C}$:

$$
\begin{array}{ccc}
F(A) \otimes F(B) \xrightarrow{s'} F(B) \otimes F(A) \\
{\scriptstyle \Phi}\downarrow \qquad\qquad\qquad \downarrow{\scriptstyle \Phi} \\
F(A \otimes B) \xrightarrow[F(s)]{} F(B \otimes A)
\end{array}
$$

Remark 3. *In this paper, we use the terminology of categorical semi-ring for the notion of a symmetric bimonoidal category given in Definition 2.1.1 in [27], see also [19,28]. Moreover, some of*

the examples we are studying are a particular case that has the structure of a distributive symmetric monoidal category given in Definition 2.3.1 in [19]. For general information about monoidal categories, we refer the reader to [29–31]. Some interesting relations between monoidal categories and classifying spaces are given in [32] and, for new advances in the study of monoidal categories, you can see [33–35].

In the following paragraphs, we include the notion of a free strict monoidal category generated by a category.

For any given category $\mathbf{C}$, the *free monoidal category* over $\mathbf{C}$, denoted as $\mathbf{Free}(\mathbf{C})$, is given as follows:

- Its objects are finite sequences $(A_1, \cdots, A_n)$ of objects in $\mathbf{C}$. The empty sequence is also considered, and it is taken as the unit object I.
- Consider two objects $(A_1, \cdots, A_m)$ and $(B_1, \cdots, B_n)$ in $\mathbf{Free}(\mathbf{C})$. If $m = n$, then a morphism $(A_1, \cdots, A_n) \xrightarrow{f} (B_1, \cdots, B_n)$ consists of a sequence $f = (f_1, ..., f_n)$ where $f_i : A_i \to B_i$ is a morphism in $\mathbf{C}$, for all $i \in \{1, ..., n\}$. If $n \neq m$, then there are no morphisms between $(A_1, \cdots, A_m)$ and $(B_1, \cdots, B_n)$.

The tensor product of two finite sequences $A = (A_1, \cdots, A_m)$ and $B = (B_1, \cdots, B_n)$ is defined as their concatenation

$$A \otimes B := (A_1, \cdots A_m, B_1, \cdots B_n)$$

Analogously, the tensor product of two morphisms in $\mathbf{Free}(\mathbf{C})$ is given by the corresponding concatenation of morphisms in $\mathbf{C}$. One can straightforwardly check that $\mathbf{Free}(\mathbf{C})$ is a strict monoidal category. In addition, observe that we have a canonical functor

$$in : \mathbf{C} \hookrightarrow \mathbf{Free}(\mathbf{C}),$$

which is, actually, a full embedding.

Any monoidal category that is isomorphic (in the monoidal sense) to a free monoidal category will be also called free monoidal category.

The free monoidal category $\mathbf{Free}(\mathbf{C})$ has the following universal property:

Proposition 1. *Let $F : \mathbf{C} \to \mathbf{D}$ be a functor between a category $\mathbf{C}$ and a strict monoidal category $\mathbf{D} = (\mathbf{D}, \otimes_D, I_D)$. Then, there exists a $(2,0)$-monoidal functor, which is unique up to isomorphism, making commutative the following triangle:*

$$
\begin{array}{ccc}
\mathbf{C} & \xrightarrow{\ F\ } & \mathbf{D} \\
{\scriptstyle in}\Big\uparrow\Big\downarrow & \nearrow & \\
\mathbf{Free}(\mathbf{C}) & \scriptstyle ext(F) &
\end{array}
$$

Proof. The functor is defined as $ext(F)(I) := I_D$ and $ext(F)(A) := F(A)$, for every object A in $\mathbf{C}$. In addition, $ext(F)(A_1, A_2) := F(A_1) \otimes_D F(A_2)$ for every object (A_1, A_2) of length 2. Now, if $ext(F)$ is defined for sequences of objects of length $n - 1$, then we may inductively define

$$ext(F)(A_1, \cdots, A_n) := ext(F)(A_1, ..., A_{n-1}) \otimes_D F(A_n).$$

F is similarly defined for morphisms. The rest of the proof is lengthy but straightforward. □

Remark 4. *In the context of monoidal categories, it is interesting to take into account the following result, whose proof can be found in [36]:*

MacLane's Theorem. *Given a monoidal category $\mathbf{C}$, there exist a strict monoidal category $\mathbf{C}'$ together with a monoidal equivalence $F : \mathbf{C} \to \mathbf{C}'$. Similarly, given a braided (resp. symmetric)*

monoidal category **C**, *there exist a strict braided (resp. symmetric) monoidal category* **C′** *with a braided (resp. symmetric) monoidal equivalence* $F : \mathbf{C} \to \mathbf{C'}$.

2.3. Extending Monoidal Structures to Presheaves

Consider $\mathbf{C} = (\mathbf{C}, \otimes, I)$ a small monoidal category. We will see that the monoidal structure on **C** can be extended to the category of presheaves $\mathbf{Sets}^{\mathbf{C}^{op}}$. Indeed, first of all, observe that we have a diagram given by two rectangles

$$
\begin{array}{ccccc}
\mathbf{C} \times \mathbf{C} & \!\!=\!\!=\!\!=\!\!=\!\! & \mathbf{C} \times \mathbf{C} & \xrightarrow{\;\otimes\;} & \mathbf{C} \\
{\scriptstyle \mathbf{Y} \times \mathbf{Y}} \downarrow & & {\scriptstyle \mathbf{Y}} \downarrow & & {\scriptstyle \mathbf{Y}} \downarrow \\
\mathbf{Sets}^{\mathbf{C}^{op}} \times \mathbf{Sets}^{\mathbf{C}^{op}} & \xrightarrow[\;\bar{\times}\;]{} & \mathbf{Sets}^{(\mathbf{C} \times \mathbf{C})^{op}} & \dashrightarrow[\;L^{\mathbf{Y} \circ \otimes}\;] & \mathbf{Sets}^{\mathbf{C}^{op}}
\end{array}
$$

where $\bar{\times} : \mathbf{Sets}^{C^{op}} \times \mathbf{Sets}^{C^{op}} \to \mathbf{Sets}^{(\mathbf{C} \times \mathbf{C})^{op}}$ is the natural functor induced by the product of sets: $(X \bar{\times} Y)(c, c') := X(c) \times Y(c')$; and the functor $L^{\mathbf{Y} \circ \otimes}$ is the colimit-preserving extension of the composite functor $Z := \mathbf{Y} \circ \otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{op}}$ given after using Theorem 1. Composing these two rectangles and denoting as $\widetilde{\otimes}$ the composite $L^{\mathbf{Y} \circ \otimes} \circ \bar{\times}$, we obtain a commutative diagram:

$$
\begin{array}{ccc}
\mathbf{C} \times \mathbf{C} & \xrightarrow{\;\otimes\;} & \mathbf{C} \\
{\scriptstyle \mathbf{Y} \times \mathbf{Y}} \downarrow & & {\scriptstyle \mathbf{Y}} \downarrow \\
\mathbf{Sets}^{\mathbf{C}^{op}} \times \mathbf{Sets}^{\mathbf{C}^{op}} & \xrightarrow[\;\widetilde{\otimes}\;]{} & \mathbf{Sets}^{\mathbf{C}^{op}}
\end{array}
$$

Moreover, we can also consider the Yoneda object $\widetilde{I} := \mathbf{Y}(I)$.

Theorem 2. $(\mathbf{Sets}^{\mathbf{C}^{op}}, \widetilde{\otimes}, \widetilde{I})$ *is a monoidal category and the Yoneda embedding,* $\mathbf{Y} : (\mathbf{C}, \otimes, I) \to (\mathbf{Sets}^{\mathbf{C}^{op}}, \widetilde{\otimes}, \widetilde{I})$, *is* $(2, 0)$-*monoidal.*

Proof. In order to prove that $(\mathbf{Sets}^{\mathbf{C}^{op}}, \widetilde{\otimes}, \widetilde{I})$ is a monoidal category one has just to take into account that any presheaf on **C** is naturally isomorphic to a colimit of representable presheaves (see Corollary 1) and that $(\mathbf{C}, \otimes, I)$ is monoidal. We leave the details to the reader. The Yoneda embedding is $(2, 0)$-monoidal by the above commutative diagram relating $\otimes$ and $\widetilde{\otimes}$; moreover, $\widetilde{I} = \mathbf{Y}(I)$ by definition. $\square$

Now, we want to extend Theorem 1 for the special case $\mathcal{E} = \mathbf{Sets}^{\mathbf{C}^{op}}$ when we consider monoidal structures. In order to do this, we need some previous results. Assume $\mathcal{I}$ and $\mathcal{J}$ are small categories and $X : \mathcal{I} \to \mathbf{Sets}$, $Y : \mathcal{J} \to \mathbf{Sets}$ functors. Then, the existence of a natural isomorphism is well known:

$$
\operatorname{colim}_{i \in \mathcal{I}} X_i \times \operatorname{colim}_{j \in \mathcal{J}} Y_j \cong \operatorname{colim}_{(i,j) \in \mathcal{I} \times \mathcal{J}} (X_i \times Y_j).
$$

By how the functor $\bar{\times} : \mathbf{Sets}^{\mathbf{C}^{op}} \times \mathbf{Sets}^{\mathbf{C}^{op}} \to \mathbf{Sets}^{(\mathbf{C} \times \mathbf{C})^{op}}$ is defined, it is straightforward to check that this property also holds when we have functors $X : \mathcal{I} \to \mathbf{Sets}^{\mathbf{C}^{op}}$ and $Y : \mathcal{J} \to \mathbf{Sets}^{\mathbf{C}^{op}}$:

$$
(\operatorname{colim}_{i \in \mathcal{I}} X_i) \bar{\times} (\operatorname{colim}_{j \in \mathcal{J}} Y_j) \cong \operatorname{colim}_{(i,j) \in \mathcal{I} \times \mathcal{J}} (X_i \bar{\times} Y_j).
$$

Indeed, for any object (c, c') in the product category $\mathbf{C} \times \mathbf{C}$, we have

$$
\begin{aligned}
[(\mathrm{colim}_{i \in \mathcal{I}} X_i) \bar{\times} (\mathrm{colim}_{j \in \mathcal{J}} Y_j)](c, c') &\cong (\mathrm{colim}_{i \in \mathcal{I}} X_i)(c) \times (\mathrm{colim}_{j \in \mathcal{J}} Y_j)(c') \\
&\cong \mathrm{colim}_{i \in \mathcal{I}} X_i(c) \times \mathrm{colim}_{j \in \mathcal{J}} Y_j(c') \\
&\cong \mathrm{colim}_{(i,j) \in \mathcal{I} \times \mathcal{J}} (X_i(c) \times Y_j(c')) \\
&\cong \mathrm{colim}_{(i,j) \in \mathcal{I} \times \mathcal{J}} ((X_i \bar{\times} Y_j)(c, c')) \\
&\cong [\mathrm{colim}_{(i,j) \in \mathcal{I} \times \mathcal{J}} ((X_i \bar{\times} Y_j)](c, c').
\end{aligned}
$$

For more results related to Theorem 2, we refer the reader to [37,38].

Taking into account this latter natural isomorphism, we have the following result:

Proposition 2. *Consider $\mathcal{I}$, $\mathcal{J}$ small categories and functors $X : \mathcal{I} \to \mathbf{Sets}^{\mathbf{C}^{op}}$, $Y : \mathcal{J} \to \mathbf{Sets}^{\mathbf{C}^{op}}$. Then, there exists a natural isomorphism*

$$
(\mathrm{colim}_{i \in \mathcal{I}} X_i) \widetilde{\otimes} (\mathrm{colim}_{j \in \mathcal{J}} Y_j) \cong \mathrm{colim}_{(i,j) \in \mathcal{I} \times \mathcal{J}} (X_i \widetilde{\otimes} Y_j).
$$

Proof. Considering the previous comments and taking into account that the functor $L^{\mathbf{Y} \circ \otimes} : \mathbf{Sets}^{(\mathbf{C} \times \mathbf{C})^{op}} \to \mathbf{Sets}^{\mathbf{C}^{op}}$ preserves colimits, we have:

$$
\begin{aligned}
(\mathrm{colim}_{i \in \mathcal{I}} X_i) \widetilde{\otimes} (\mathrm{colim}_{j \in \mathcal{J}} Y_j) &= L^{\mathbf{Y} \circ \otimes} ((\mathrm{colim}_{i \in \mathcal{I}} X_i) \bar{\times} (\mathrm{colim}_{j \in \mathcal{J}} Y_j)) \\
&\cong L^{\mathbf{Y} \circ \otimes} (\mathrm{colim}_{(i,j) \in \mathcal{I} \times \mathcal{J}} (X_i \bar{\times} Y_j)) \\
&\cong \mathrm{colim}_{(i,j) \in \mathcal{I} \times \mathcal{J}} (L^{\mathbf{Y} \circ \otimes} (X_i \bar{\times} Y_j)) \\
&\cong \mathrm{colim}_{(i,j) \in \mathcal{I} \times \mathcal{J}} (X_i \widetilde{\otimes} Y_j).
\end{aligned}
$$

$\square$

Remember that, by using Theorem 1 when $\mathcal{E} = \mathbf{Sets}^{\mathbf{C}^{op}}$, for any given functor $Z : \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{op}}$, there exists a colimit preserving functor $L^Z : \mathbf{Sets}^{\mathbf{C}^{op}} \to \mathbf{Sets}^{\mathbf{C}^{op}}$ such that $L^Z \circ \mathbf{Y} = Z$. When Z is $(2, 0)$-monoidal, we can say more about L^Z.

Theorem 3. *Let $Z : \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{op}}$ be a $(2, 0)$-monoidal functor. Then, the colimit-preserving functor $L^Z : \mathbf{Sets}^{\mathbf{C}^{op}} \to \mathbf{Sets}^{\mathbf{C}^{op}}$, which makes commutative the diagram*

is $(2, 0)$-monoidal. Moreover, up to isomorphism, it is the unique colimit preserving functor making this diagram commutative.

Proof. Consider X and Y presheaves over $\mathbf{C}$. We know by Corollary 1 that, up to natural isomorphism, any presheaf is the colimit of representable presheaves. Therefore, we can suppose that

$$
X \cong \mathrm{colim}_{i \in \mathcal{I}} \mathbf{Y}(c_i) \quad \text{and} \quad Y \cong \mathrm{colim}_{j \in \mathcal{J}} \mathbf{Y}(c_j).
$$

Therefore,

$$
\begin{aligned}
L^Z(X \widetilde{\otimes} Y) &\cong L^Z((\mathrm{colim}_{i \in \mathcal{I}} \, \mathbf{Y}(c_i)) \widetilde{\otimes} (\mathrm{colim}_{j \in \mathcal{J}} \, \mathbf{Y}(c_j))) \\
&\cong L^Z(\mathrm{colim}_{(i,j) \in \mathcal{I} \times \mathcal{J}} \, (\mathbf{Y}(c_i) \widetilde{\otimes} \mathbf{Y}(c_j))) \\
&\cong \mathrm{colim}_{(i,j) \in \mathcal{I} \times \mathcal{J}} \, L^Z(\mathbf{Y}(c_i) \widetilde{\otimes} \mathbf{Y}(c_j)) \\
&\cong \mathrm{colim}_{(i,j) \in \mathcal{I} \times \mathcal{J}} \, L^Z(\mathbf{Y}(c_i \otimes c_j)) \\
&\cong \mathrm{colim}_{(i,j) \in \mathcal{I} \times \mathcal{J}} \, Z(c_i \otimes c_j) \\
&\cong \mathrm{colim}_{(i,j) \in \mathcal{I} \times \mathcal{J}} \, (Z(c_i) \widetilde{\otimes} Z(c_j)) \\
&\cong (\mathrm{colim}_{i \in \mathcal{I}} \, Z(c_i)) \widetilde{\otimes} (\mathrm{colim}_{j \in \mathcal{J}} \, Z(c_j)) \\
&\cong (\mathrm{colim}_{i \in \mathcal{I}} \, L^Z(\mathbf{Y}(c_i))) \widetilde{\otimes} (\mathrm{colim}_{j \in \mathcal{J}} \, L^Z(\mathbf{Y}(c_j))) \\
&\cong L^Z(\mathrm{colim}_{i \in \mathcal{I}} \, \mathbf{Y}(c_i)) \widetilde{\otimes} L^Z(\mathrm{colim}_{j \in \mathcal{J}} \, \mathbf{Y}(c_j)) \\
&\cong L^Z(X) \widetilde{\otimes} L^Z(Y).
\end{aligned}
$$

Moreover, $L^Z(\widetilde{I}) = L^Z(\mathbf{Y}(I)) = Z(I) \cong \widetilde{I}$. Checking that the coherence diagrams are indeed commutative is lengthy but straightforward. □

Remark 5. *Observe that Theorem 3 above can be easily generalized for any $(2,0)$-monoidal functor $Z : \mathbf{C} \to \mathcal{E}$, where $\mathcal{E} = (\mathcal{E}, \otimes_\mathcal{E}, I_\mathcal{E})$ is a monoidal cocomplete category satisfying the additional condition that, for any two diagrams $X : \mathcal{I} \to \mathcal{E}$, $Y : \mathcal{J} \to \mathcal{E}$, there exists a natural isomorphism*

$$
(\mathrm{colim}_{i \in \mathcal{I}} X_i) \otimes_\mathcal{E} (\mathrm{colim}_{j \in \mathcal{J}} Y_j) \cong \mathrm{colim}_{(i,j) \in \mathcal{I} \times \mathcal{J}} (X_i \otimes_\mathcal{E} Y_j).
$$

We have an extension functor $L^Z : \mathbf{Sets}^{\mathbf{C}^{op}} \to \mathcal{E}$ that is $(2,0)$-monoidal. Moreover, up to isomorphism, L^Z is the unique colimit preserving functor making the following diagram commutative

3. Augmented Semi-Simplicial Sets

In this section, we will deal with the combinatorial objects we are interested in for this work, the augmented semi-simplicial sets. As we are about to see, the category of augmented semi-simplicial sets is defined as the category of presheaves on a certain small category.

3.1. Basic Notions and Results

Consider the small category $\mathbf{\Gamma}$ whose objects are non-empty totally ordered sets $[p] = \{0 < \cdots < p\}$ for $p \geq 0$ and whose morphisms are the strictly increasing maps $[p] \to [q]$. We can add the empty set $\varnothing = [-1]$ to this category together with all the strictly increasing maps. The new augmented category will be denoted by $\mathbf{\Gamma}_+$.

We can consider $\mathbf{Sets}^{\mathbf{\Gamma}_+^{op}}$ and $\mathbf{Sets}^{\mathbf{\Gamma}_+}$, the presheaf categories on $\mathbf{\Gamma}_+$ and on $\mathbf{\Gamma}_+^{op}$, respectively. Taking $\mathcal{E} = \mathbf{Sets}^{\mathbf{\Gamma}_+^{op}}$, $\mathbf{C} = \mathbf{\Gamma}^+$, and considering the notation $X \widetilde{\rhd} Y$ for $L^Y(X)$ given in Definition 1 we have the following action functors:

$$
\mathbf{Sets}^{\mathbf{\Gamma}_+^{op}} \times (\mathbf{Sets}^{\mathbf{\Gamma}_+^{op}})^{\mathbf{\Gamma}_+} \to \mathbf{Sets}^{\mathbf{\Gamma}_+^{op}}, \quad (X, Y) \mapsto X \widetilde{\rhd} Y,
$$

$$
(\mathbf{Sets}^{\mathbf{\Gamma}_+^{op}})^{\mathbf{\Gamma}_+} \times (\mathbf{Sets}^{\mathbf{\Gamma}_+^{op}})^{\mathbf{\Gamma}_+} \to (\mathbf{Sets}^{\mathbf{\Gamma}_+^{op}})^{\mathbf{\Gamma}_+}, \quad (Y, Z) \mapsto Y \widetilde{\rhd} Z.
$$

We obviously have the corresponding Yoneda embeddings:

$$
\mathbf{Y} : \mathbf{\Gamma}_+ \to \mathbf{Sets}^{\mathbf{\Gamma}_+^{op}}, \quad \mathbf{Y}^{op} : \mathbf{\Gamma}_+^{op} \to \mathbf{Sets}^{\mathbf{\Gamma}_+}.
$$

For any object $[n]$ in $\mathbf{\Gamma}_+$, we will denote as $\mathbf{\Gamma}_+[n]$ the representable presheaf

$$\mathbf{\Gamma}_+[n] := \mathbf{Y}([n]) = \mathrm{Hom}_{\mathbf{\Gamma}_+}(-, [n]) \in \mathbf{Sets}^{\mathbf{\Gamma}_+^{op}}.$$

Analogously, we will consider the notation

$$\mathbf{\Gamma}_+^{op}[n] := \mathbf{Y}^{op}([n]) = \mathrm{Hom}_{\mathbf{\Gamma}_+}([n], -) \in \mathbf{Sets}^{\mathbf{\Gamma}_+}.$$

Definition 5. *Any presheaf on $\mathbf{\Gamma}_+$*

$$X : \mathbf{\Gamma}_+^{op} \to \mathbf{Sets}$$

will be called augmented semi-simplicial set or $\mathbf{\Gamma}_+^{op}$-set. Analogously, any presheaf on $\mathbf{\Gamma}_+^{op}$, $Z :$ $\mathbf{\Gamma}_+ \to \mathbf{Sets}$, will be called augmented co-semi-simplicial set or $\mathbf{\Gamma}_+$-set.

We point out that giving an augmented semi-simplicial set $X \in \mathbf{Sets}^{\mathbf{\Gamma}_+^{op}}$ is equivalent to giving a collection of sets $\{X_n\}_{n \geq -1}$ together with a collection of set maps $d_i^n : X_n \to X_{n-1}$ ($n \geq 0$ and $0 \leq i \leq n$) satisfying

$$d_i^n \circ d_j^{n+1} = d_{j-1}^n \circ d_i^{n+1}, \text{ if } i < j.$$

Moreover, giving an arrow $f : X \to \bar{X}$ in $\mathbf{Sets}^{\mathbf{\Gamma}_+^{op}}$ (i.e., a natural transformation) is equivalent to giving a collection of set maps $\{f_n : X_n \to \bar{X}_n\}_{n \geq -1}$ such that

$$f_{n-1} \circ d_i^n = \bar{d}_i^{n-1} \circ f_n,$$

for $n \geq 0$ and $0 \leq i \leq n$.

There is a similar description for the category of augmented co-semi-simplicial sets by just reversing the arrows in the above representation.

In this work, we also consider the subcategory of finite sets $\mathbf{Sets}_{\mathrm{fin}}$ and the corresponding category of presheaves $(\mathbf{Sets}_{\mathrm{fin}})^{\mathbf{\Gamma}_+^{op}}$.

A semi-simplicial set $X \in \mathbf{Sets}^{\mathbf{\Gamma}_+^{op}}$ is said to have *finite dimension* if there is $k \in \mathbb{N}_+$ such that $X_i = \varnothing$ for every $i > k$. Given a non-empty finite dimensional semi-simplicial set X, we denote

$$\dim(X) := \min\{k \mid X_i = \varnothing \text{ for all } i > k, i \in \mathbb{N}_+\}$$

For the empty semi-simplicial set, we set $\dim(\varnothing) := -\infty$

The subcategory of finite dimensional semi-simplicial sets is denoted by $(\mathbf{Sets}^{\mathbf{\Gamma}_+^{op}})_{\mathrm{fd}}$. A semi-simplicial set $X \in \mathbf{Sets}^{\mathbf{\Gamma}_+^{op}}$ is said to be *finite* if it has finite dimension and for every $i \in \mathbb{N}_+$, X_i is a finite set. The subcategory of finite semi-simplicial sets is denoted by $(\mathbf{Sets}^{\mathbf{\Gamma}_+^{op}})_{\mathrm{fin}}$.

3.2. Products and Coproducts

The usual product $\times$ and coproduct $\sqcup$ of sets induce, in a natural way, two symmetric monoidal category structures in $\mathbf{Sets}$ with unit objects the singleton 1 and the empty set $\varnothing$, respectively.

Given $X, Y \in \mathbf{Sets}^{\mathbf{\Gamma}_+^{op}}$, we have the product $X \times Y$ and the coproduct $X \sqcup Y$, which are straightforwardly induced by the corresponding coproduct and product in $\mathbf{Sets}$. For any given set $A \in \mathbf{Sets}$, we can consider the constant $\mathbf{\Gamma}_+^{op}$-set, Γ_+^A, defined as $(\Gamma_+^A)_k = A$, for all $k \geq -1$. In particular, we are interested in the constant $\mathbf{\Gamma}_+^{op}$-sets, $\Gamma_+^\varnothing$ and Γ_+^1. It is plain to check the following natural isomorphisms:

$$X \sqcup (Y \sqcup Z) \cong (X \sqcup Y) \sqcup Z,$$

$$\Gamma_+^\varnothing \sqcup X \cong X \cong X \sqcup \Gamma_+^\varnothing,$$

$$X \times (Y \times Z) \cong (X \times Y) \times Z,$$

$$\Gamma_+^1 \times X \cong X \cong X \times \Gamma_+^1,$$

$$X \times Y \cong Y \times X, \quad X \sqcup Y \cong Y \sqcup X.$$

Note that the subcategory $\mathbf{Sets}_{\mathrm{fin}}$ is closed by finite sums and products and that $\varnothing, 1$ are finite sets.

Therefore, we obtain the following result:

Proposition 3. *The sum and the product induce the following structures:*

(i) $(\mathbf{Sets}^{\Gamma_+^{op}}, \sqcup, \Gamma_+^{\varnothing})$ *and* $(\mathbf{Sets}^{\Gamma_+^{op}}, \times, \Gamma_+^1)$ *are symmetric monoidal categories.*

(ii) $((\mathbf{Sets}_{fin})^{\Gamma_+^{op}}, \sqcup, \Gamma_+^{\varnothing})$ *and* $((\mathbf{Sets}_{fin})^{\Gamma_+^{op}}, \times, \Gamma_+^1)$ *are symmetric monoidal categories.*

(iii) *the canonical inclusions*

$$((\mathbf{Sets}_{fin})^{\Gamma_+^{op}}, \sqcup, \Gamma_+^{\varnothing}) \to (\mathbf{Sets}^{\Gamma_+^{op}}, \sqcup, \Gamma_+^{\varnothing})$$

$$((\mathbf{Sets}_{fin})^{\Gamma_+^{op}}, \times, \Gamma_+^1) \to (\mathbf{Sets}^{\Gamma_+^{op}}, \times, \Gamma_+^1)$$

are $(2,0)$*-monoidal functors.*

Moreover, we also have the natural isomorphism relating the two tensor products:

$$X \times (Y \sqcup Z) \cong (X \times Y) \sqcup (X \times Z).$$

We also observe that, for every $[n] \in \Gamma_+$, one has the evaluation functor

$$\mathrm{ev}_n : \mathbf{Sets}^{\Gamma_+^{op}} \to \mathbf{Sets}, \quad X \mapsto X_n$$

that preserves the (co)product, and the units. In particular, we can consider $(2,0)$-monoidal functors

$$\mathrm{ev}_n : (\mathbf{Sets}^{\Gamma_+^{op}}, \sqcup, \Gamma_+^{\varnothing}) \to (\mathbf{Sets}, \sqcup, \varnothing),$$

$$\mathrm{ev}_n : (\mathbf{Sets}^{\Gamma_+^{op}}, \times, \Gamma_+^1) \to (\mathbf{Sets}, \times, 1).$$

Remark 6. *It is obvious that* $\Gamma_+^{\varnothing} \in (\mathbf{Sets}^{\Gamma_+^{op}})_{fin}$. *Therefore,* $((\mathbf{Sets}^{\Gamma_+^{op}})_{fin}, \sqcup, \Gamma_+^{\varnothing})$ *is a symmetric monoidal category. On the other hand,* $\Gamma_+^1 \notin (\mathbf{Sets}^{\Gamma_+^{op}})_{fin}$, *indicating that* $((\mathbf{Sets}^{\Gamma_+^{op}})_{fin}, \times, \Gamma_+^1)$ *is not a monoidal category. However, taking the full subcategory* $(\mathbf{Sets}^{\Gamma_+^{op}})_{fin} \cup \{\Gamma_+^1\}$, *one has that* $((\mathbf{Sets}^{\Gamma_+^{op}})_{fin} \cup \{\Gamma_+^1\}, \times, \Gamma_+^1)$ *is a monoidal category.*

3.3. Join, Cone, and Suspension Constructions in Augmented Semi-Simplicial Sets

Now, we will consider special constructions in Γ_+^{op}-sets, or augmented semi-simplicial sets. We first observe that the small category Γ_+ has a symmetric monoidal category structure induced by the coproduct (actually, the ordinal sum)

$$[p] \sqcup [q] := [p + q + 1]$$

having $[-1]$ as a unit object.

We define the join of Γ_+^{op}-sets as the natural extension of this ordinal sum to $\mathbf{Sets}^{\Gamma_+^{op}}$ (see Section 2.3). We obtain the *join functor*

$$\boxplus : \mathbf{Sets}^{\Gamma_+^{op}} \times \mathbf{Sets}^{\Gamma_+^{op}} \to \mathbf{Sets}^{\Gamma_+^{op}}.$$

It is immediate to check that, for any pair of $\mathbf{\Gamma}_+^{op}$-sets, X, Y, their join is given as

$$(X \boxplus Y)_m := \bigsqcup_{p+q=m-1} X_p \times Y_q$$

where p and q run over the integers that are greater than or equal to -1. In particular, $(X \boxplus Y)_{-1} = X_{-1} \times Y_{-1}$, $(X \boxplus Y)_0 = (X_{-1} \times Y_0) \sqcup (X_0 \times Y_{-1})$, $(X \boxplus Y)_1 = (X_{-1} \times Y_1) \sqcup (X_0 \times Y_0) \sqcup (X_1 \times Y_{-1})$ and so on.

Moreover, the operators $d_i^{X \boxplus Y}$ of $X \boxplus Y$ are naturally defined from the operators d_k^X and d_l^Y of X and Y, respectively. The definition of $\boxplus$ on morphisms is straightforward.

For any pair $X, Y \in \mathbf{Sets}^{\mathbf{\Gamma}_+^{op}}$, we have that the functors

$$X \boxplus (-), \ (-) \boxplus Y : \mathbf{Sets}^{\mathbf{\Gamma}_+^{op}} \to \mathbf{Sets}^{\mathbf{\Gamma}_+^{op}}$$

preserve colimits. Furthermore, if $X, Y, Z \in \mathbf{Sets}^{\mathbf{\Gamma}_+^{op}}$, then there exist canonical natural isomorphisms

$$X \boxplus (Y \boxplus Z) \cong (X \boxplus Y) \boxplus Z,$$

$$\Gamma_+[-1] \boxplus X \cong X \cong X \boxplus \Gamma_+[-1],$$

$$X \boxplus Y \cong Y \boxplus X.$$

Corollary 2. *The category* $\mathbf{Sets}^{\mathbf{\Gamma}_+^{op}}$*, together with the join functor* $\boxplus$*, the unit object* $\Gamma_+[-1]$*, and the natural isomorphisms above, is a symmetric monoidal category. Moreover, the Yoneda embedding*

$$\mathbf{Y} : (\mathbf{\Gamma}_+, \sqcup, [-1]) \to (\mathbf{Sets}^{\mathbf{\Gamma}_+^{op}}, \boxplus, \Gamma_+[-1])$$

is $(2, 0)$*-monoidal.*

Proof. This is a direct consequence of Theorem 2. □

Remark 7. *Note that* $((\mathbf{Sets}^{\mathbf{\Gamma}_+^{op}})_{fin}, \boxplus, \Gamma_+[-1])$ *is a monoidal subcategory of* $((\mathbf{Sets}_{fin})^{\mathbf{\Gamma}_+^{op}}, \boxplus, \Gamma_+[-1])$*, which is a monoidal subcategory of* $(\mathbf{Sets}^{\mathbf{\Gamma}_+^{op}}, \boxplus, \Gamma_+[-1])$*. If* $X, Y \in (\mathbf{Sets}^{\mathbf{\Gamma}_+^{op}})_{fin}$*, one has that:*

$$\dim(X \boxplus Y) = (\dim(X) + 1) + (\dim(Y) + 1) - 1 = \dim(X) + \dim(Y) + 1.$$

Corollary 3. *Given* $n, m \in \mathbb{N}_+$*, the following isomorphism holds true*

$$\Gamma_+[n] \boxplus \Gamma_+[m] \cong \Gamma_+[n + m + 1].$$

We have the *left-cone functor* $\mathrm{Con}_l := \Gamma_+[0] \boxplus (-)$ and the *right-cone functor* $\mathrm{Con}_r := (-) \boxplus \Gamma_+[0]$:

$$\mathrm{Con}_l, \mathrm{Con}_r : \mathbf{Sets}^{\mathbf{\Gamma}_+^{op}} \to \mathbf{Sets}^{\mathbf{\Gamma}_+^{op}}$$

$$\mathrm{Con}_l(X) = \Gamma_+[0] \boxplus X,$$

$$\mathrm{Con}_r(X) = X \boxplus \Gamma_+[0].$$

These functors make the following squares commutative:

$$
\begin{array}{ccc}
\mathbf{\Gamma}_+ & \xrightarrow{[0] \sqcup (-)} & \mathbf{\Gamma}_+ \\
\mathbf{Y} \downarrow & & \downarrow \mathbf{Y} \\
\mathbf{Sets}^{\mathbf{\Gamma}_+^{op}} & \xrightarrow[\mathrm{Con}_l]{} & \mathbf{Sets}^{\mathbf{\Gamma}_+^{op}}
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{\Gamma}_+ & \xrightarrow{(-) \sqcup [0]} & \mathbf{\Gamma}_+ \\
\mathbf{Y} \downarrow & & \downarrow \mathbf{Y} \\
\mathbf{Sets}^{\mathbf{\Gamma}_+^{op}} & \xrightarrow[\mathrm{Con}_r]{} & \mathbf{Sets}^{\mathbf{\Gamma}_+^{op}}.
\end{array}
$$

Corollary 4. *Con$_l$ and Con$_r$ satisfy*

$$Con_l(\Gamma_+[k]) = \Gamma_+[k+1] = Con_r(\Gamma_+[k])$$

for all $k \geq -1$.

Now, consider the augmented 0-sphere

$$S_+[0] = \Gamma_+[1] \setminus \{\iota_1\}$$

where ι_1 denotes the identity of $[1] \in \mathbf{\Gamma}_+$.

Then, as a particular case of the join construction, and similarly to the case of the cones, we have the *augmented left-suspension* functor $Sus_l = S_+[0] \boxplus (-)$ and the *augmented right-suspension* functor $Sus_r = (-) \boxplus S_+[0]$ (see Figures 1, 2):

$$Sus_l, Sus_r : \mathbf{Sets}^{\mathbf{\Gamma}_+^{op}} \to \mathbf{Sets}^{\mathbf{\Gamma}_+^{op}}$$

$$Sus_l(X) = S_+[0] \boxplus X$$

$$Sus_r(X) = X \boxplus S_+[0]$$
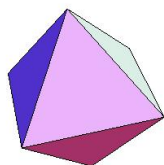
Figure 1. The join $S_+[0] \boxplus S_+[0] \boxplus S_+[0]$.
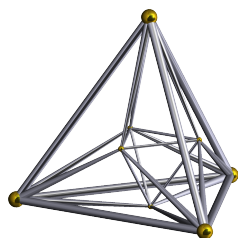
Figure 2. The join $S_+[0] \boxplus S_+[0] \boxplus S_+[0] \boxplus S_+[0]$.

3.4. Action Functors and Triangle Products

We can consider $\mathbf{Sets}^{\mathbf{\Gamma}_+^{op}}$ and $\mathbf{Sets}^{\mathbf{\Gamma}_+}$, the presheaf categories on $\mathbf{\Gamma}_+$ and $\mathbf{\Gamma}_+^{op}$, respectively. Remember that taking $\mathcal{E} = \mathbf{Sets}^{\mathbf{\Gamma}_+^{op}}$, $\mathbf{C} = \mathbf{\Gamma}^+$, and, considering the notation $X \widetilde{\triangleright} Y$ given in Definition 1, we have the action functors:

$$\mathbf{Sets}^{\mathbf{\Gamma}_+^{op}} \times (\mathbf{Sets}^{\mathbf{\Gamma}_+^{op}})^{\mathbf{\Gamma}_+} \to \mathbf{Sets}^{\mathbf{\Gamma}_+^{op}}, \quad (X, Y) \mapsto X \widetilde{\triangleright} Y$$

$$(\mathbf{Sets}^{\mathbf{\Gamma}_+^{op}})^{\mathbf{\Gamma}_+} \times (\mathbf{Sets}^{\mathbf{\Gamma}_+^{op}})^{\mathbf{\Gamma}_+} \to (\mathbf{Sets}^{\mathbf{\Gamma}_+^{op}})^{\mathbf{\Gamma}_+}, \quad (Y, Z) \mapsto Y \widetilde{\triangleright} Z$$

We observe that the cylinder and the barycentric subdivision of a semi-simplicial set can be obtained by using action functors and some particular co-semi-simplicial objects $Z \in (\mathbf{Sets}^{\mathbf{\Gamma}_+^{op}})^{\mathbf{\Gamma}_+}$.

Remark 8. *If $\mathbf{\Gamma}_+|_{[-1],[0]}$ denotes the full subcategory of $\mathbf{\Gamma}_+$ whose objects are $[-1]$ and $[0]$, then it is plain to check that, up to isomorphism, $\mathbf{\Gamma}_+$ together with the coproduct (ordinal sum) is isomorphic to the free monoidal category over $\mathbf{\Gamma}_+|_{[-1],[0]}$. This fact will be crucial for the next reasoning.*

Let $\mathbf{C}$ be any category. Then, in order to consider a functor $F : \Gamma_+|_{[-1],[0]} \to \mathbf{C}$, one has just to give a morphism $f : X_{-1} \to X_0$ in $\mathbf{C}$. Moreover, if $\mathbf{C} = (\mathbf{C}, \otimes, I)$ is monoidal, one can take the particular case $X_{-1} \cong I$. The following result is a consequence of Remark 8 and Proposition 1.

Corollary 5. *Let $\mathbf{C} = (\mathbf{C}, \otimes, I)$ be a monoidal category and $F : \Gamma_+|_{[-1],[0]} \to \mathbf{C}$ the functor given by a morphism $f : I \to X$. Then, there exists a $(2, 0)$-monoidal functor extending F:*

$$
\begin{array}{ccc}
\Gamma_+|_{[-1],[0]} & \xrightarrow{F} & \mathbf{C} \\
{\scriptstyle in}\downarrow & \nearrow & \\
\Gamma_+ & {\scriptstyle ext(F)} &
\end{array}
$$

which is given by $ext(F)[n] = \otimes_0^n X$, $ext(F)[-1] = I$ and $ext(F)([-1] \to [0]) = f$. Moreover, $ext(F) : (\Gamma_+, \sqcup, [-1]) \to (\mathbf{C}, \otimes, I)$ is, up to isomorphism, the unique $(2, 0)$-monoidal extension of F.

Definition 6. *Let $\mathbf{C} = (\mathbf{C}, \otimes, I)$ be a monoidal category, $f : I \to X$ a morphism in $\mathbf{C}$ and $F : \Gamma_+|_{[-1],[0]} \to \mathbf{C}$ the functor induced by f. Then, the $(2, 0)$-monoidal extension $ext(F) : (\Gamma_+, \sqcup, [-1]) \to (\mathbf{C}, \otimes, I)$ is said to be the* co-semi-simplicial *extension of f and it is denoted by $\otimes_f X (= ext(F))$. When there is a unique morphism f from I to X, we can use the notation $\otimes X$, deleting f in $\otimes_f X$, and we say the $\otimes X$ is the* co-semi-simplicial *extension of X.*

Definition 7. *Let $Z : \Gamma_+ \to \mathbf{Sets}^{\Gamma_+^{op}}$ be a functor. If $X \in \mathbf{Sets}^{\Gamma_+^{op}}$, then the object $X \widetilde{\rhd} Z$ is said to be the* tilde-triangle product *of X and Z. If $f : \Gamma_+[-1] \to Y$ is a morphism in $\mathbf{Sets}^{\Gamma_+^{op}}$, then $X \rhd_f Y := X \widetilde{\rhd} (\otimes_f Y)$ is said to be the* f-triangle product *of X and Y. When there is a unique morphism f from $\Gamma_+[-1]$ to Y, $X \rhd_f Y$ is simply denoted as $X \rhd Y$, and it is said to be the* triangle product *of X and Y.*

The following result is a direct consequence of Theorem 3. We recall again that we are using the particular notation $(-)\widetilde{\rhd}Z$ for the construction $L^Z(-)$:

Corollary 6. *Let $Z : (\Gamma_+, \sqcup, [-1]) \to (\mathbf{Sets}^{\Gamma_+^{op}}, \boxplus, \Gamma_+[-1])$ be a $(2, 0)$-monoidal functor. Then, the colimit-preserving functor $(-)\widetilde{\rhd}Z : \mathbf{Sets}^{\Gamma_+^{op}} \to \mathbf{Sets}^{\Gamma_+^{op}}$, which makes commutative the diagram*

$$
\begin{array}{ccc}
\Gamma_+ & \xrightarrow{Z} & \mathbf{Sets}^{\Gamma_+^{op}} \\
{\scriptstyle \mathbf{Y}}\downarrow & \nearrow & \\
\mathbf{Sets}^{\Gamma_+^{op}} & {\scriptstyle (-)\widetilde{\rhd}Z} &
\end{array}
$$

is $(2, 0)$-monoidal. In particular, for all $X, Y \in \mathbf{Sets}^{\Gamma_+^{op}}$, we have

$$
(X \boxplus Y)\widetilde{\rhd}Z \cong (X \widetilde{\rhd} Z) \boxplus (Y \widetilde{\rhd} Z).
$$

Moreover, up to isomorphism, it is the unique colimit preserving functor making this diagram commutative.

Remark 9. *The monoidal category $((\mathbf{Sets}^{\Gamma_+^{op}})_{\mathrm{fin}}, \boxplus, \Gamma_+[-1])$ is a monoidal subcategoy of $((\mathbf{Sets}_{\mathrm{fin}})^{\Gamma_+^{op}}, \boxplus, \Gamma_+[-1])$ which is, in turn, a monoidal subcategoy of $(\mathbf{Sets}^{\Gamma_+^{op}}, \boxplus, \Gamma_+[-1])$. Observe that:*

(i) *if $Z : (\Gamma_+, \sqcup, [-1]) \to ((\mathbf{Sets}_{\mathrm{fin}})^{\Gamma_+^{op}}, \boxplus, \Gamma_+[-1])$ is a $(2, 0)$-monoidal functor, then we obtain $(-)\widetilde{\rhd}Z : (\mathbf{Sets}_{\mathrm{fin}})^{\Gamma_+^{op}} \to (\mathbf{Sets}_{\mathrm{fin}})^{\Gamma_+^{op}}$ a $(2, 0)$-monoidal functor.*

(ii)   *if $Z : (\mathbf{\Gamma}_+, \sqcup, [-1]) \to ((\mathbf{Sets}^{\mathbf{\Gamma}_+^{op}})_{\mathrm{fin}}, \boxplus, \Gamma_+[-1])$ is a $(2,0)$-monoidal functor, then we also obtain that $(-) \widetilde{\rhd} Z : (\mathbf{Sets}^{\mathbf{\Gamma}_+^{op}})_{\mathrm{fin}} \to (\mathbf{Sets}^{\mathbf{\Gamma}_+^{op}})_{\mathrm{fin}}$ is a $(2,0)$-monoidal functor.*

4. Augmented Integer Sequences and Matrices

Once we have analyzed some nice categorical and combinatorial properties of the category of augmented semi-simplicial sets, we study the second kind of mathematical objects in this work: augmented integer sequences (and matrices). The connection between these two different worlds will become clear in the next section through the cardinal functor.

4.1. The Categories $\mathbb{Z}$, $\mathbb{N}_+$, $\mathbf{Z}$ and Some Functor Categories

We know that the set of integer numbers $\mathbb{Z}$ admits the structure of a discrete category. However, we can also consider it as a groupoid $\mathbf{Z}$ where the cardinal $|\mathrm{Hom}_{\mathbf{Z}}(n,m)| = 1$ and the unique morphism from n to m can be denoted as $m - n \colon n \to m$, for every pair of integers n, m. The sum of integers can be easily extended to a functor

$$+ : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}, \quad (n, m) \mapsto n + m.$$

Taking $+$ as a tensor product and 0 as a unit object, it is immediate to check that $(\mathbf{Z}, +, 0)$ has the structure of a strict symmetric monoidal category and also of a strict categorical group.

Remark 10. *By a categorical group, we mean a monoidal category $\mathbf{C}$ which is also a groupoid and there exists a functor $(-)^* : \mathbf{C} \to \mathbf{C}$ together with natural isomorphisms m, n and with components $m_A : A \otimes A^* \to I$, $n_A : A^* \otimes A \to I$ satisfying that, for every object A, the following diagram is commutative:*

$$
\begin{array}{ccc}
(A \otimes A^*) \otimes A & \xrightarrow{\;\;\alpha\;\;} & A \otimes (A^* \otimes A) \\
{\scriptstyle m \otimes 1}\big\downarrow & & \big\downarrow{\scriptstyle 1 \otimes n} \\
I \otimes A \xrightarrow{\;\lambda\;} A & \xleftarrow{\;\rho\;} & A \otimes I
\end{array}
$$

Roughly speaking, every object is invertible, up to natural isomorphism, with respect to the tensor product. In addition, if $\mathbf{C}$ is strict as a monoidal category and the natural isomorphisms m, n are identities, then we have that $\mathbf{C}$ is a strict categorical group.

It is plain to check that, given any small category $\mathbf{J}$, the functor category $\mathbf{Z}^{\mathbf{J}}$ has an induced strict symmetric monoidal category structure (in addition, it is a strict symmetric categorical group). We shall consider $(\mathbf{Z}^{\mathbf{J}})_{\mathrm{fin}}$ the full subcategory of $\mathbf{Z}^{\mathbf{J}}$ consisting of functors $c : \mathbf{J} \to \mathbf{Z}$ such that there exists a finite set of objects F_c satisfying that $c(j) = 0$ for all $j \in \mathbf{J} \setminus F_c$.

Now, if $\mathbb{N}$ denotes that the set of natural numbers (0 is also included as a natural number), take the discrete category $\mathbb{N}_+ = \mathbb{N} \cup \{-1\}$. This category can be considered as a subcategory of both $\mathbf{\Gamma}_+$ and $\mathbf{\Gamma}_+^{op}$ through the (inclusion) functor $n \to [n]$. Observe that the category $\mathbb{N}_+$ is self-dual, that is, $\mathbb{N}_+ = \mathbb{N}_+^{op}$.

Given a functor $c \in (\mathbf{Z}^{\mathbb{N}_+^{op}})_{\mathrm{fin}}$ with $c \neq 0$, we will call *dimension of c* the integer

$$\dim(c) := \min\{k \mid c_i = 0 \text{ for all } i > k, i \in \mathbb{N}_+\}$$

and, for $c = 0$, we take

$$\dim(c) := -\infty$$

Taking into account that $\mathbb{N}_+^{op}$ is a discrete category, we have that the Yoneda embedding $\mathbf{Y} : \mathbb{N}_+ \to \mathbf{Sets}^{\mathbb{N}_+^{op}}$ associated with $\mathbb{N}_+$ (note that this functor is a restriction of the Yoneda functor $\mathbf{Y} : \mathbf{\Gamma}_+ \to \mathbf{Sets}^{\mathbf{\Gamma}_+^{op}}$) induces, by applying the cardinal operator to the corresponding

hom-sets, a functor $\mathbf{y} : \mathbb{N}_+ \to \mathbf{Z}^{\mathbb{N}_+^{op}}$ satisfying $\mathbf{y}(n)(j) = \delta_{n,j}$, for all $n, j \in \mathbb{N}_+$. Here, $\delta_{n,j}$ denotes the Kronecker delta

$$\delta_{n,j} = \begin{cases} 1, & n = j \\ 0, & n \neq j \end{cases}$$

Therefore, $\mathbf{y}(n) \in (\mathbf{Z}^{\mathbb{N}_+^{op}})_{\text{fin}}$ and $\dim(\mathbf{y}(n)) = n$, for all $n \in \mathbb{N}_+$. This way, we actually have a functor

$$\mathbf{y} : \mathbb{N}_+ \to (\mathbf{Z}^{\mathbb{N}_+^{op}})_{\text{fin}}.$$

In the following result, we may think that the monoidal category $\mathbf{C}$ is $\mathbf{Z}$. Thus, we will keep the notation $+$ for the tensor product and 0 for the unit.

Proposition 4. *Let* $(\mathbf{C}, +, 0)$ *be a strict symmetric categorical group. If* $Z : \mathbb{N}_+ \to \mathbf{C}$ *is any functor, then there exists, up to natural isomorphism, a unique* $(2, 0)$-*monoidal functor*

$$(-) \cdot Z \colon (\mathbf{Z}^{\mathbb{N}_+^{op}})_{\text{fin}} \to \mathbf{C}, \quad c \mapsto c \cdot Z$$

making the following diagram commutative:

$$
\begin{array}{ccc}
\mathbb{N}_+ & \xrightarrow{\ Z\ } & \mathbf{C} \\
{\scriptstyle \mathbf{y}}\Big\downarrow & \nearrow_{(-)\cdot Z} & \\
(\mathbf{Z}^{\mathbb{N}_+^{op}})_{\text{fin}} & &
\end{array}
$$
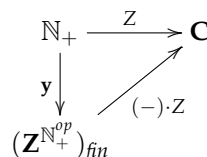
Proof. Given any integer $\lambda \in \mathbb{Z}$ and any object $a \in \mathbf{C}$, we can define, in a natural way, another object $\lambda a \in \mathbf{C}$ (similarly $a\lambda$) given from the tensor product $+$ in $(\mathbf{C}, +, 0)$ and the existence of the inverse object. Namely,

$$\lambda a := \begin{cases} \underbrace{a + \ldots + a}_{\lambda \ times}, & \text{if } \lambda > 0 \\ 0, & \text{if } \lambda = 0 \\ \underbrace{a^* + \ldots + a^*}_{-\lambda \ times}, & \text{if } \lambda < 0 \end{cases}$$

Now, consider a functor $c \in (\mathbf{Z}^{\mathbb{N}_+^{op}})_{\text{fin}}$. If we denote $c_k = c(k) \in \mathbb{Z}$ and $Z_k := Z(k) \in \mathbf{C}$, then we define

$$((-) \cdot Z)(c) = c \cdot Z := \sum_{k \in \mathbb{N}_+} c_k Z_k,$$

which is well defined since it is a finite sum. Note that, if $c \neq 0$, we have

$$((-) \cdot Z)(c) = c \cdot Z = \sum_{k=-1}^{\dim(c)} c_k Z_k.$$

In addition, $(-) \cdot Z$ is accordingly defined for morphisms so that we obtain a functor. A straightforward inspection proves that such a functor makes the diagram commutative, and it is $(2, 0)$-monoidal. Moreover, it is unique up to natural isomorphism, satisfying these properties. $\square$

Remark 11. *We point out that the functor*

$$(-) \cdot Z : ((\mathbf{Z}^{\mathbb{N}_+^{op}})_{\text{fin}}, +, 0) \to (\mathbf{C}, +, 0)$$

is always $(2, 0)$-*monoidal regardless of whether the functor* Z *is* $(2, 0)$-*monoidal or not. This means that* $((\mathbf{Z}^{\mathbb{N}_+^{op}})_{\text{fin}}, +, 0)$ *is the free (strict) symmetric categorical group in the discrete category* $\mathbb{N}_+$.

Remark 12. *We have a dual version of Proposition [4]. If we take a functor $Z : \mathbb{N}_+^{op} \to \mathbf{C}$, then there exists, up to natural isomorphism, a unique $(2,0)$-monoidal functor $Z \cdot (-) : (\mathbf{Z}^{\mathbb{N}_+})_{fin} \to \mathbf{C}$, $c \mapsto Z \cdot c$, which is an extension of Z.*

Definition 8. *An augmented integer sequence is a functor*

$$a : \mathbb{N}_+^{op} \to \mathbb{Z}.$$

More generally, given any category $\mathbf{C}$, we call augmented sequence in $\mathbf{C}$ any functor $a : \mathbb{N}_+^{op} \to \mathbf{C}$. We will denote an augmented sequence a by means of a row matrix

$$a = (a_{-1}\ \ a_0\ \ a_1\ \ a_2\ \ \cdots).$$

However, in some cases, this row matrix will be denoted by (using commas): $a = (a_{-1}, a_0, a_1, a_2, \cdots)$.

Analogously, an augmented integer co-sequence is a functor $b : \mathbb{N}_+ \to \mathbb{Z}$. More generally, for a given category $\mathbf{C}$, an augmented co-sequence in $\mathbf{C}$ is a functor $b : \mathbb{N}_+ \to \mathbf{C}$. We will denote an augmented co-sequence b by means of a column matrix

$$b = \begin{pmatrix} b_{-1} \\ b_0 \\ b_1 \\ \vdots \end{pmatrix}$$

or $b = (b_{-1}\ b_0\ b_1\ \cdots)^T$, where T denotes the transposition operator.

Remark 13. *Observe that, in the definition above, we have given two different names to the same mathematical object (remember that $\mathbb{N}_+$ is self-dual). Nevertheless, associated with each name, we are using a different notation, which will be crucial when we consider matrix products later on.*

Definition 9. *An augmented integer matrix is a functor $U : \mathbb{N}_+ \times \mathbb{N}_+^{op} \to \mathbb{Z}$. In general, given a category $\mathbf{C}$, an augmented matrix with entries in $\mathbf{C}$ is just a functor $U : \mathbb{N}_+ \times \mathbb{N}_+^{op} \to \mathbf{C}$. An augmented matrix U will be denoted by its usual form*

$$U = \begin{pmatrix} U_{-1,-1} & U_{-1,0} & U_{-1,1} & \cdots \\ U_{0,-1} & U_{0,0} & U_{0,1} & \cdots \\ U_{1,-1} & U_{1,0} & U_{1,1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Remark 14. *For any given categories $\mathbf{A}, \mathbf{B}, \mathbf{C}$, there are canonical isomorphisms $\mathbf{A} \times \mathbf{B} \cong \mathbf{B} \times \mathbf{A}$ and $\mathbf{C}^{\mathbf{A} \times \mathbf{B}} \cong (\mathbf{C}^{\mathbf{A}})^{\mathbf{B}}$. Therefore, we have induced isomorphisms*

$$(\mathbf{C}^{\mathbb{N}_+})^{\mathbb{N}_+^{op}} \cong \mathbf{C}^{\mathbb{N}_+ \times \mathbb{N}_+^{op}} \cong \mathbf{C}^{\mathbb{N}_+^{op} \times \mathbb{N}_+} \cong (\mathbf{C}^{\mathbb{N}_+^{op}})^{\mathbb{N}_+}.$$

As a consequence, any augmented matrix $U : \mathbb{N}_+ \times \mathbb{N}_+^{op} \to \mathbf{C}$ may be considered as an object in any of the categories above.

Definition 10. *Let $(\mathbf{C}, +, 0)$ be a strict symmetric categorical group. The following functors will be called dot-products:*

$$(-) \cdot (-) : (\mathbf{Z}^{\mathbb{N}_+^{op}})_{fin} \times \mathbf{C}^{\mathbb{N}_+} \to \mathbf{C}, \quad (c, Z) \mapsto c \cdot Z := \sum_{k \in \mathbb{N}_+} c_k Z_k$$

$$(-) \cdot (-) : \mathbf{C}^{\mathbb{N}_+^{op}} \times (\mathbf{Z}^{\mathbb{N}_+})_{fin} \to \mathbf{C}, \quad (Y, b) \mapsto Y \cdot b := \sum_{k \in \mathbb{N}_+} Y_k b_k$$

Remark 15. *If* $(\mathbf{C}, +, 0)$ *is a strict symmetric categorical group and we consider the subcategory* $(\mathbf{C}^{\mathbb{N}_+})_{fin}$ *of those functors* $Z \colon \mathbb{N}_+ \to \mathbf{C}$ *satisfying that there is* $k \in \mathbb{N}_+$ *with* $Z_l = 0$ *for* $l \geq k$ *(* $(\mathbf{C}^{\mathbb{N}_+^{op}})_{fin}$ *is analogously defined), then we also have natural dot-products:*

$$(-) \cdot (-) \colon \mathbf{Z}^{\mathbb{N}_+^{op}} \times (\mathbf{C}^{\mathbb{N}_+})_{fin} \to \mathbf{C}, \quad (c, Z) \mapsto c \cdot Z := \sum_{k \in \mathbb{N}_+} c_k Z_k$$

$$(-) \cdot (-) \colon (\mathbf{C}^{\mathbb{N}_+^{op}})_{fin} \times \mathbf{Z}^{\mathbb{N}_+} \to \mathbf{C}, \quad (Y, b) \mapsto Y \cdot b := \sum_{k \in \mathbb{N}_+} Y_k b_k$$

In this case, the functors $(-) \cdot Z \colon (\mathbf{Z}^{\mathbb{N}_+^{op}}, +, 0) \to (\mathbf{C}, +, 0)$, $Y \cdot (-) \colon (\mathbf{Z}^{\mathbb{N}_+^{op}}, +, 0) \to (\mathbf{C}, +, 0)$ *are also* $(2, 0)$-*monoidal.*

4.2. Multiplication (Dot-Product) of Augmented Matrices

Now, we see how matrix multiplication is induced by the categorical group structure. Consider $(\mathbf{C}, +, 0) = (\mathbf{Z}, +, 0)$; if $a \in (\mathbf{Z}^{\mathbb{N}_+^{op}})_{fin}$ and $b \colon \mathbb{N}_+ \to \mathbf{C}$ is any functor, then we have that they are of the form

$$a = (a_{-1}, a_0, a_1, \cdots a_n, 0, 0, 0, \cdots) \quad \text{and} \quad b = (b_{-1}, b_0, b_1, b_2, b_3, \cdots)^T$$

Therefore, $a \cdot b = \sum_{i=-1}^{+\infty} a_i b_i$ is well defined and we have an induced bifunctor

$$(-) \cdot (-) \colon (\mathbf{Z}^{\mathbb{N}_+^{op}})_{fin} \times \mathbf{Z}^{\mathbb{N}_+} \to \mathbf{Z}, \quad (a, b) \mapsto a \cdot b$$

Similarly, we have a bifunctor

$$(-) \cdot (-) \colon \mathbf{Z}^{\mathbb{N}_+^{op}} \times (\mathbf{Z}^{\mathbb{N}_+})_{fin} \to \mathbf{Z}, \quad (c, d) \mapsto c \cdot d.$$

Taking into account the transposition isomorphism $(-)^T \colon \mathbf{Z}^{\mathbb{N}_+^{op}} \overset{\cong}{\to} \mathbf{Z}^{\mathbb{N}_+}$, we have that the composition induced by the identity on the first variable and the transposition on the second variable induces the scalar (or inner) product:

$$\langle -, - \rangle \colon (\mathbf{Z}^{\mathbb{N}_+^{op}})_{fin} \times (\mathbf{Z}^{\mathbb{N}_+^{op}})_{fin} \to \mathbf{Z}.$$

Namely, if $a = (a_{-1}, a_0, a_1, \cdots a_n, 0 \cdots)$ and $b = (b_{-1}, b_0, b_1, \cdots b_m, 0 \cdots)$, then

$$\langle a, b \rangle = a \cdot b^T = \sum_{i=-1}^{\min\{n,m\}} a_i b_i.$$

It is easy to check that one has the following canonical extended bifunctors of the dot-product:

$$(-) \cdot (-) \colon (\mathbf{Z}^{\mathbb{N}_+^{op}})_{fin} \times (\mathbf{Z}^{\mathbb{N}_+^{op}})^{\mathbb{N}_+} \to \mathbf{Z}^{\mathbb{N}_+^{op}}, (a, B) \to a \cdot B,$$

$$(-) \cdot (-) \colon \mathbf{Z}^{\mathbb{N}_+^{op}} \times ((\mathbf{Z}^{\mathbb{N}_+})_{fin})^{\mathbb{N}_+^{op}} \to \mathbf{Z}^{\mathbb{N}_+^{op}}, (a, B) \to a \cdot B,$$

$$(-) \cdot (-) \colon (\mathbf{Z}^{\mathbb{N}_+^{op}})^{\mathbb{N}_+} \times (\mathbf{Z}^{\mathbb{N}_+})_{fin} \to \mathbf{Z}^{\mathbb{N}_+}, (A, b) \to A \cdot b,$$

$$(-) \cdot (-) \colon ((\mathbf{Z}^{\mathbb{N}_+^{op}})_{fin})^{\mathbb{N}_+} \times \mathbf{Z}^{\mathbb{N}_+} \to \mathbf{Z}^{\mathbb{N}_+}, (A, b) \to A \cdot b,$$

$$(a \cdot B)_j = \sum_{k \in \mathbb{N}_+} a_k B_{k,j}, \quad (A \cdot b)_i = \sum_{k \in \mathbb{N}_+} A_{i,k} b_k.$$

$$(-) \cdot (-) \colon ((\mathbf{Z}^{\mathbb{N}_+^{op}})_{fin})^{\mathbb{N}_+} \times (\mathbf{Z}^{\mathbb{N}_+})^{\mathbb{N}_+^{op}} \to \left(\mathbf{Z}^{\mathbb{N}_+}\right)^{\mathbb{N}_+^{op}}, (A, B) \to A \cdot B,$$

$$(-) \cdot (-) \colon (\mathbf{Z}^{\mathbb{N}_+^{op}})^{\mathbb{N}_+} \times \left((\mathbf{Z}^{\mathbb{N}_+})_{fin}\right)^{\mathbb{N}_+^{op}} \to (\mathbf{Z}^{\mathbb{N}_+^{op}})^{\mathbb{N}_+}, (A, B) \to A \cdot B,$$

$$(A \cdot B)_{i,j} = \sum_{k \in \mathbb{N}_+} A_{i,k} B_{k,j}.$$

4.3. Join of Numerical Augmented Sequences: Cone and Suspension

We note that $\mathbf{Z}^{\mathbb{N}_+^{op}}$ has the structure of a ring $(\mathbf{Z}^{\mathbb{N}_+^{op}}, +, \times)$, which is induced by the ring structure of $(\mathbf{Z}, +, \times)$ by pointwise operation. That is, for all $a, b \in \mathbf{Z}^{\mathbb{N}_+^{op}}$, their sum and product are given as

$$(a + b)_i = a_i + b_i, \quad (a \times b)_i = a_i \times b_i, \quad i \in \mathbb{N}_+.$$

For $a, b \in \mathbf{Z}$, we will also use the notation $a \times b = ab$.

However, we may consider a new symmetric monoidal structure on $\mathbf{Z}^{\mathbb{N}_+^{op}}$. This is given by the join product:

Definition 11. *The join product of $a, b \in \mathbf{Z}^{\mathbb{N}_+^{op}}$, denoted as $a \boxplus b$, is given by the following formula:*

$$(a \boxplus b)_m := \sum_{p+q=m-1} a_p b_q, \quad p, q \in \mathbb{N}_+.$$

In this case, the unit object is given as $\mathbf{1}_{-1}$ where $(\mathbf{1}_{-1})_i = \delta_{-1,i}$ is the Kronecker delta. In this work, for $k \in \mathbb{N}_+$, $\mathbf{1}_k$ will denote the augmented sequence given by $(\mathbf{1}_k)_i = \delta_{k,i}$.

Proposition 5. *The category $\mathbf{Z}^{\mathbb{N}_+^{op}}$ equipped with the join product $\boxplus$ and the unit object $\mathbf{1}_{-1}$ has the structure of a strict symmetric monoidal category. Moreover, the induced functor $\mathbf{y} \colon (\mathbb{N}_+, \sqcup, [-1])$ $\to (\mathbf{Z}^{\mathbb{N}_+^{op}}, \boxplus, \mathbf{1}_{-1})$ is $(2,0)$-monoidal.*

Remark 16. *It is an obvious fact that $((\mathbf{Z}^{\mathbb{N}_+^{op}})_{fin}, \boxplus, \mathbf{1}_{-1})$ is a monoidal subcategory of $(\mathbf{Z}^{\mathbb{N}_+^{op}}, \boxplus, \mathbf{1}_{-1})$. If $a, b \in (\mathbf{Z}^{\mathbb{N}_+^{op}})_{fin}$, then, by our definition of dimension, one has:*

$$\dim(a \boxplus b) = (\dim(a) + 1) + (\dim(b) + 1) - 1 = \dim(a) + \dim(b) + 1.$$

If $a, b \in \mathbf{Z}^{\mathbb{N}_+^{op}}$ are fixed, then we easily obtain functors

$$a \boxplus (-), (-) \boxplus b \colon \mathbf{Z}^{\mathbb{N}_+^{op}} \to \mathbf{Z}^{\mathbb{N}_+^{op}}$$

given by $c \mapsto a \boxplus c$ and $c \mapsto c \boxplus b$, respectively.

Now, for each $k \in \mathbb{Z}$, we define an operator in the category $\mathbf{Z}^{\mathbb{N}_+^{op}}$ (actually, a functor)

$$\mathrm{D}_k : \mathbf{Z}^{\mathbb{N}_+^{op}} \to \mathbf{Z}^{\mathbb{N}_+^{op}}, \quad c \to \mathrm{D}_k(c)$$

as follows: given $c \in \mathbf{Z}^{\mathbb{N}_+^{op}}$ and $i \in \mathbb{N}_+$,

If $k \geq 0$, then $(\mathrm{D}_k(c))_i = c_{i+k}$,
If $k \leq 0$, then:

$$(\mathrm{D}_k(c))_i = \begin{cases} c_{i+k}, & \text{if } i + k \geq -1, \\ 0, & \text{if } i + k < -1. \end{cases}$$

Definition 12. *For a given $b \in \mathbf{Z}^{\mathbb{N}_+^{op}}$, we define $R(b) \in \left((\mathbf{Z}^{\mathbb{N}_+})_{\text{fin}} \right)^{\mathbb{N}_+^{op}}$, the shifting of b, by the formula $(R(b))_i = D_{-(i+1)}(b)$, for all $i \in \mathbb{N}_+$. We may also see it as the matrix*

$$
R(b) = \begin{pmatrix} D_0(b) \\ D_{-1}(b) \\ D_{-2}(b) \\ \vdots \end{pmatrix}
$$

This construction naturally gives rise to a functor

$$
R : \mathbf{Z}^{\mathbb{N}_+^{op}} \to \left((\mathbf{Z}^{\mathbb{N}_+})_{\text{fin}} \right)^{\mathbb{N}_+^{op}}
$$

which satisfies, in a natural way, the commutativity given in the next result.

Proposition 6. *The diagram*

$$
\begin{array}{ccc}
\mathbf{Z}^{\mathbb{N}_+^{op}} \times \mathbf{Z}^{\mathbb{N}_+^{op}} & \xrightarrow{\ id \times R\ } & \mathbf{Z}^{\mathbb{N}_+^{op}} \times \left((\mathbf{Z}^{\mathbb{N}_+})_{\text{fin}} \right)^{\mathbb{N}_+^{op}} \\
& (-) \boxplus (-) \searrow \quad \swarrow (-)\cdot(-) & \\
& \mathbf{Z}^{\mathbb{N}_+^{op}} &
\end{array}
$$

is commutative. In other words, given $a, b \in \mathbf{Z}^{\mathbb{N}_+^{op}}$, we have the equalities

$$
a \boxplus b = a \cdot R(b) = b \cdot R(a).
$$

Now, we present an interesting construction: the cone of a sequence.

Definition 13. *We define the cone of $c \in \mathbf{Z}^{\mathbb{N}_+^{op}}$ as the sequence $con(c) \in \mathbf{Z}^{\mathbb{N}_+^{op}}$ defined as $con(c) := c + D_{-1}(c)$. That is to say,*

$$
con(c)_i = \begin{cases} c_i + c_{i-1}, & \text{if } i \geq 0 \\ c_{-1}, & \text{if } i = -1. \end{cases}
$$

Manifestly, we obtain the *cone functor* $con : \mathbf{Z}^{\mathbb{N}_+^{op}} \to \mathbf{Z}^{\mathbb{N}_+^{op}}$. Taking into account the results above, the next consequence is straightforward to check.

Proposition 7. *Consider $\mathbf{c} = con(\mathbf{1}_{-1}) = \mathbf{1}_{-1} + \mathbf{1}_0 \in \mathbf{Z}^{\mathbb{N}_+^{op}}$. Then, the cone functor is related to the join and the dot product through the following formula:*

$$
con(b) = \mathbf{c} \boxplus b = b \boxplus \mathbf{c} = b \cdot R(\mathbf{c}).
$$

Remark 17. *Iterating the cone functor and bearing in mind the identity $con(b) = b \cdot R(\mathbf{c})$, which is given in the result above, we obtain that $con^k(b) = b \cdot (R(\mathbf{c}))^k$ ($k \geq 0$). This way, $con^k(b)$ is completely determined by the sequence of matrices $(R(\mathbf{c}))^0 = id$, $(R(\mathbf{c}))^1 = R(\mathbf{c})$, $(R(\mathbf{c}))^2, \dots$*

$$
R(\mathbf{c}) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (R(\mathbf{c}))^2 = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 2 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \dots
$$

It is also simple to check that the cone functor has an inverse functor

$$con^{-1} : \mathbf{Z}^{\mathbb{N}_+^{op}} \to \mathbf{Z}^{\mathbb{N}_+^{op}}, \quad con^{-1}(b) = b \cdot R(\mathbf{c})^{-1},$$

where

$$R(\mathbf{c})^{-1} = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & \cdots \\ 0 & 1 & -1 & 1 & -1 & 1 & -1 & \cdots \\ 0 & 0 & 1 & -1 & 1 & -1 & 1 & \cdots \\ 0 & 0 & 0 & 1 & -1 & 1 & -1 & \cdots \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is the inverse of $R(\mathbf{c})$.

Consequently, the inverse of the cone functor may be expressed by the equation

$$(con^{-1}(b))_k = \sum_{i=-1}^{k} (-1)^{k-i} b_i$$

where $b \in \mathbf{Z}^{\mathbb{N}_+^{op}}$ and $k \in \mathbb{N}_+$.

Next, we define the Euler q-characteristic of any $c \in (\mathbf{Z}^{\mathbb{N}_+^{op}})_{\text{fin}}$.

Definition 14. *Consider a sequence $c \in (\mathbf{Z}^{\mathbb{N}_+^{op}})_{fin}$ and a rational number $q \in \mathbb{Q}$. Then, if $c \neq 0$, the Euler q-characteristic of c is the rational number defined as*

$$E_q(c) = \sum_{i=-1}^{dim(c)} (-q)^i c_i.$$

When $c = 0$, we take $E_q(c) = 0$.

Remark 18. *If $c \neq 0$ and $q = -1$, then we obtain $sum(c)$, the sum of all elements of c :*

$$E_{-1}(c) = \sum_{i=-1}^{dim(c)} c_i = sum(c).$$

When $q = 1$, we have the Euler characteristic:

$$E_1(c) = \sum_{i=-1}^{dim(c)} (-1)^i c_i = E(c),$$

and, for $q = \frac{1}{2}$, we obtain

$$E_{\frac{1}{2}}(c) = \sum_{i=-1}^{dim(c)} \frac{(-1)^i}{2^i} c_i.$$

Remark 19. *We have that $E_1(con(c)) = 0$ for $c \in (\mathbf{Z}^{\mathbb{N}_+^{op}})_{fin}$. Now, if $(\mathbf{Z}^{\mathbb{N}_+^{op}})_{fin,1}$ denotes the full subcategory of $(\mathbf{Z}^{\mathbb{N}_+^{op}})_{fin}$ consisting of those sequences $c \in (\mathbf{Z}^{\mathbb{N}_+^{op}})_{fin}$ with Euler characteristics equal to zero, then the (restricted) cone functor $con : (\mathbf{Z}^{\mathbb{N}_+^{op}})_{fin} \to (\mathbf{Z}^{\mathbb{N}_+^{op}})_{fin,1}$ is an isomorphism. Its inverse functor*

$$con^{-1} : (\mathbf{Z}^{\mathbb{N}_+^{op}})_{fin,1} \to (\mathbf{Z}^{\mathbb{N}_+^{op}})_{fin}$$

can be given for $b \neq 0$ by the expressions (see the formula above for the inverse $(con^{-1}(b))_k$)

$$(con^{-1}(b))_k = \sum_{i=k}^{dim(b)-1} (-1)^{i-k} b_{i+1} = \sum_{i=-1}^{k} (-1)^{i+k} b_i$$

since $E_1(b) = 0$.

 Note that a finite augmented sequence not satisfying the Euler characteristic condition is the cone of an infinite-dimensional augmented sequence. For instance,

$$con((1, -1, 1, -1, 1, -1, \cdots)) = (1, 0, 0, 0, \cdots)$$

where the first sequence contains negative integers.

 We are also interested in the suspension of a sequence.

Definition 15. *The* suspension *of a sequence $c \in \mathbf{Z}^{\mathbb{N}_+^{op}}$ c is the sequence $sus(c) \in \mathbf{Z}^{\mathbb{N}_+^{op}}$ defined as $sus(c) := c + 2D_{-1}(c)$. In other words,*

$$(sus(c))_i = \begin{cases} c_i + 2c_{i-1}, & \text{if } i \geq 0 \\ c_{-1}, & \text{if } i = -1. \end{cases}$$

 There is an obvious induced *suspension functor* sus : $\mathbf{Z}^{\mathbb{N}_+^{op}} \to \mathbf{Z}^{\mathbb{N}_+^{op}}$, which is straightforwardly related to the join and the dot product.

Proposition 8. *Consider $\mathbf{s} = sus(\mathbf{1}_{-1}) = \mathbf{1}_{-1} + \mathbf{2}_0 \in \mathbf{Z}^{\mathbb{N}_+^{op}}$. Then, the suspension functor sus$: \mathbf{Z}^{\mathbb{N}_+^{op}} \to \mathbf{Z}^{\mathbb{N}_+^{op}}$ is related to the join and the dot product through the formula*

$$sus(b) = \mathbf{s} \boxplus b = b \boxplus \mathbf{s} = b \cdot R(\mathbf{s}).$$

Remark 20. *Taking the iteration of the suspension functor and the relation given in the result above, we obtain that $sus^k(b) = b \cdot (R(\mathbf{s}))^k$ ($k \geq 0$). That is, as in the case of the cone, the iteration of the suspension is completely determined by the sequence of matrices $(R(\mathbf{s}))^0 = id$, $(R(\mathbf{s}))^1 = R(\mathbf{s})$, $(R(\mathbf{s}))^2, \dots$*

$$R(\mathbf{s}) = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (R(\mathbf{s}))^2 = \begin{pmatrix} 1 & 4 & 4 & 0 & 0 & \cdots \\ 0 & 1 & 4 & 4 & 0 & \cdots \\ 0 & 0 & 1 & 4 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \cdots$$

 The suspension functor has an inverse functor

$$sus^{-1} : \mathbf{Z}^{\mathbb{N}_+^{op}} \to \mathbf{Z}^{\mathbb{N}_+^{op}}, \quad sus^{-1}(b) = b \cdot (R(\mathbf{s}))^{-1},$$

where

$$(R(\mathbf{s}))^{-1} = \begin{pmatrix} 1 & -2 & 4 & -8 & 16 & -32 & 64 & \cdots \\ 0 & 1 & -2 & 4 & -8 & 16 & -32 & \cdots \\ 0 & 0 & 1 & -2 & 4 & -8 & 16 & \cdots \\ 0 & 0 & 0 & 1 & -2 & 4 & -8 & \cdots \\ 0 & 0 & 0 & 0 & 1 & -2 & 4 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots \end{pmatrix}$$

Hence, one has

$$(\text{sus}^{-1}(b))_k = \sum_{i=-1}^{k} (-2)^{k-i} b_i$$

for any $b \in \mathbf{Z}^{\mathbb{N}_+^{op}}$ and $k \in \mathbb{N}_+$.

Remark 21. *We want to mention that $E_{\frac{1}{2}}(sus(c)) = 0$ for any $c \in (\mathbf{Z}^{\mathbb{N}_+^{op}})_{fin}$. Consider $(\mathbf{Z}^{\mathbb{N}_+^{op}})_{fin,\frac{1}{2}}$ the full subcategory of $(\mathbf{Z}^{\mathbb{N}_+^{op}})_{fin}$ consisting of those sequences having a Euler $\frac{1}{2}$-characteristic equal to zero. Then, the (restricted) suspension functor $sus : (\mathbf{Z}^{\mathbb{N}_+^{op}})_{fin} \to (\mathbf{Z}^{\mathbb{N}_+^{op}})_{fin,\frac{1}{2}}$ is an isomorphism. Taking into account that $E_{\frac{1}{2}}(b) = 0$, for $b \in (\mathbf{Z}^{\mathbb{N}_+^{op}})_{fin,\frac{1}{2}}$, the functor*

$$sus^{-1} : (\mathbf{Z}^{\mathbb{N}_+^{op}})_{fin,\frac{1}{2}} \to (\mathbf{Z}^{\mathbb{N}_+^{op}})_{fin}$$

can also be expressed as

$$
\begin{aligned}
(sus^{-1}(b))_k = \sum_{i=-1}^{k} (-2)^{k-i} b_i &= \sum_{i=k}^{dim(b)-1} (-1)^{i-k} \left(\tfrac{1}{2}\right)^{i-k+1} b_{i+1} \\
&= \sum_{i=k}^{dim(b)-1} (-1)^{i+k} 2^{k-i-1} b_{i+1}
\end{aligned}
$$

for $b \neq 0$.

Like the case of the cone functor, a finite augmented sequence not satisfying a Euler $\frac{1}{2}$-characteristic condition is the suspension of an infinite-dimensional augmented sequence. For instance,

$$sus((1,1,1,-2,4,-8,16,\cdots,(-2)^{n-2},\cdots,)) = (1,3,3,0,\cdots)$$

where the first sequence contains negative integers.

4.4. Actions of Augmented Matrices on Augmented Sequences of Integers

To finish this section, we consider actions of sequences and matrices. We first establish the augmented binomial matrix $\text{bin} \in ((\mathbf{Z}^{\mathbb{N}_+^{op}})_{\text{fin}})^{\mathbb{N}_+}$ defined as

$$\text{bin}_{i,j} = \binom{i+1}{j+1}, \quad i,j \in \mathbb{N}_+$$

and its inverse matrix $\text{bin}^{-1} \in ((\mathbf{Z}^{\mathbb{N}_+^{op}})_{\text{fin}})^{\mathbb{N}_+}$ given as

$$\text{bin}_{i,j}^{-1} = (-1)^{i-j} \binom{i+1}{j+1}, \quad i,j \in \mathbb{N}_+.$$

Definition 16. *Given a sequence $a \in (\mathbf{Z}^{\mathbb{N}_+^{op}})_{fin}$ and a matrix $B \in (\mathbf{Z}^{\mathbb{N}_+^{op}})^{\mathbb{N}_+}$, the action of B on a is defined by the formula*

$$a \widetilde{\triangleright} B := (a \cdot bin^{-1}) \cdot B.$$

The resulting sequence $a \widetilde{\triangleright} B$ is also said to be the tilde-triangle product of a and B. This construction gives rise to an action functor

$$(-) \widetilde{\triangleright} (-) : (\mathbf{Z}^{\mathbb{N}_+^{op}})_{fin} \times (\mathbf{Z}^{\mathbb{N}_+^{op}})^{\mathbb{N}_+} \to \mathbf{Z}^{\mathbb{N}_+^{op}}.$$

We point out that we also have the identity $a \widetilde{\triangleright} \text{bin} = (a \cdot \text{bin}^{-1}) \cdot \text{bin} = a$.

Let $\mathbb{N}_+|_{\{-1,0\}}$ denote the full subcategory of the discrete category $\mathbb{N}_+$ whose objects are -1 and 0. Then, it is plain to check that, up to isomorphism, $\mathbb{N}_+$ together with the coproduct

$$p \sqcup q := p + q + 1,$$

is isomorphic to the free monoidal category over $\mathbb{N}_+|_{\{-1,0\}}$.

If **C** is any category, then, in order to consider a functor $F : \mathbb{N}_+|_{\{-1,0\}} \to \mathbf{C}$, it is only needed to give two objects X_{-1}, X_0 in **C**. Moreover, if $\mathbf{C} = (\mathbf{C}, \otimes, I)$ is monoidal, one can consider the particular case $X_{-1} \cong I$. The following result is a direct consequence of Proposition 1.

Corollary 7. *Let* $\mathbf{C} = (\mathbf{C}, \otimes, I)$ *be a monoidal category and* $X \in \mathbf{C}$. *Then, there exists a* $(2,0)$-*monoidal functor extending* $F : \mathbb{N}_+|_{\{-1,0\}} \to \mathbf{C}$, $F(-1) = I$, $F(0) = X$:

$$
\begin{array}{ccc}
\mathbb{N}_+|_{\{-1,0\}} & \xrightarrow{\;F\;} & \mathbf{C} \\
{\scriptstyle in}\downarrow\cap & \nearrow & \\
\mathbb{N}_+ & {\scriptstyle ext(F)} &
\end{array}
$$

which is given by $ext(F)(n) = \otimes_0^n X$, *for* $n \geq 0$, *and* $ext(F)(-1) = I$. *Moreover,* $ext(F) \colon (\mathbb{N}+, \sqcup, -1) \to (\mathbf{C}, \otimes, I)$ *is, up to isomorphism, the unique* $(2,0)$-*monoidal extension of* F.

Specializing $\mathbf{C} = (\mathbf{C}, \otimes, I) = (\mathbf{Z}^{\mathbb{N}_+^{op}}, \boxplus, \mathbf{1}_{-1})$ and $b \in \mathbf{Z}^{\mathbb{N}_+^{op}}$, we obtain the notion of augmented matrix extension of b:

Definition 17. *Consider the functor* $F : \mathbb{N}_+|_{\{-1,0\}} \to \mathbf{Z}^{\mathbb{N}_+^{op}}$, $F(-1) = \mathbf{1}_{-1}$ $F(0) = b$. *The* $(2,0)$-*monoidal extension* $ext(F) \colon (\mathbb{N}_+, \sqcup, -1) \to (\mathbf{Z}^{\mathbb{N}_+^{op}}, \boxplus, \mathbf{1}_{-1})$ *is said to be the augmented matrix extension of* b, *and it is denoted as*

$$
\boxplus b = ext(F) \in (\mathbf{Z}^{\mathbb{N}_+^{op}})^{\mathbb{N}_+}.
$$

Finally, the notion of augmented matrix extension will raise the one of the triangle product:

Definition 18. *Given sequences* $a \in (\mathbf{Z}^{\mathbb{N}_+^{op}})_{fin}$ *and* $b \in \mathbf{Z}^{\mathbb{N}_+^{op}}$,

$$
a \triangleright b := a \widetilde{\triangleright} (\boxplus b)
$$

is said to be the triangle product of a *and* b.

Note that $a \triangleright b = a \widetilde{\triangleright} (\boxplus b) = (a \cdot \mathrm{bin}^{-1}) \cdot (\boxplus b) \in \mathbf{Z}^{\mathbb{N}_+^{op}}$.

5. Comparing Monoidal Categories Arising from Γ_+^{op}-Sets and Numerical Sequences

Now, we are ready to compare the category of augmented semi-simplicial finite sets and the category of augmented integer sequences. The key point is the sequential cardinal functor which applies every finite augmented semi-simplicial set to the sequence constituted by the cardinal of the set of n-simplices. This sequential cardinal functor preserves the corresponding monoidal structures. Moreover, as we will see, it preserves certain categorical semi-ring structures.

5.1. The Monoidal Categories $((\mathbf{Sets}_{\mathrm{fin}})^{\Gamma_+^{op}}, \sqcup, \Gamma_+^{\varnothing})$ and $(\mathbf{Z}^{\mathbb{N}_+^{op}}, +, 0)$ as Well as $((\mathbf{Sets}_{\mathrm{fin}})^{\Gamma_+^{op}}, \times, \Gamma_+^1)$ and $(\mathbf{Z}^{\mathbb{N}_+^{op}}, \times, 1)$

Recall that $\mathbf{Sets}_{\mathrm{fin}}$ is the category of finite sets and maps between them. Given any functor $X : \mathbf{\Gamma}_+^{op} \to \mathbf{Sets}_{\mathrm{fin}}$, we may consider the diagram

$$
\begin{array}{ccc}
\mathbb{N}_+^{op} & & \mathbf{Z} \\
\downarrow{\scriptstyle\mathrm{in}} & & \uparrow{\scriptstyle|-|} \\
\mathbf{\Gamma}_+^{op} & \xrightarrow{\quad X \quad} & \mathbf{Sets}_{\mathrm{fin}}
\end{array}
$$

where in denotes the inclusion functor and $|-|$ denotes the functor giving the cardinal of any finite set.

Definition 19. *The sequential cardinal of a $\mathbf{\Gamma}_+^{op}$- finite set, $X \in (\mathbf{Sets}_{fin})^{\Gamma_+^{op}}$, is defined as the augmented sequence:*

$$|X| : \mathbb{N}_+^{op} \to \mathbf{Z}$$

given by the composite $|X| := |-| \circ X \circ in$. To put it another way,

$$|X|_n := |X_n|, \quad n \in \mathbb{N}_+.$$

We observe that there is an induced functor $|-| : (\mathbf{Sets}_{\mathrm{fin}})^{\Gamma_+^{op}} \to \mathbf{Z}^{\mathbb{N}_+^{op}}$ where, for morphisms $f : X \to Y$, is defined as $|f| := |Y| - |X|$; that is, $|f|_n = |Y|_n - |X|_n, n \in \mathbb{N}_+$.

On the one hand, in Section 3.2, we have seen that $((\mathbf{Sets}_{\mathrm{fin}})^{\Gamma_+^{op}}, \sqcup, \Gamma_+^{\varnothing})$ and $((\mathbf{Sets}_{\mathrm{fin}})^{\Gamma_+^{op}}, \times, \Gamma_+^1)$ are monoidal categories. On the other hand, in Section 4.3, we have seen that the ring structure $(\mathbf{Z}, +, \times)$ induces the ring structure $(\mathbf{Z}^{\mathbb{N}_+^{op}}, +, \times)$ and the monoidal structures $(\mathbf{Z}^{\mathbb{N}_+^{op}}, +, 0)$, $(\mathbf{Z}^{\mathbb{N}_+^{op}}, \times, 1)$.

Proposition 9. *The functor $|-| : (\mathbf{Sets}_{fin})^{\Gamma_+^{op}} \to \mathbf{Z}^{\mathbb{N}_+^{op}}$ preserves the monoidal structures induced by coproducts and products:*

$$|-| : ((\mathbf{Sets}_{fin})^{\Gamma_+^{op}}, \sqcup, \Gamma_+^{\varnothing}) \to (\mathbf{Z}^{\mathbb{N}_+^{op}}, +, 0)$$

$$|-| : ((\mathbf{Sets}_{fin})^{\Gamma_+^{op}}, \times, \Gamma_+^1) \to (\mathbf{Z}^{\mathbb{N}_+^{op}}, \times, 1).$$

Proof. It is straightforward to check the identities:

$$|X \sqcup Y| = |X| + |Y|, \quad |X \times Y| = |X| \times |Y|$$

$$|\Gamma_+^{\varnothing}| = 0, \quad |\Gamma_+^1| = 1$$

which are left to the reader. $\square$

Definition 20. *Associated with the $\mathbf{\Gamma}_+^{op}$-sets, $\Gamma_+[n]$ and $S_+[n-1] = \Gamma_+[n] \setminus \{\iota_n\}$, where ι_n is the identity of $[n] \in \mathbf{\Gamma}_+$, we consider the sequential cardinals:*

$$\gamma_+[n] = |\Gamma_+[n]|, \quad s_+[n-1] = |S_+[n-1]|.$$

We note that $\gamma_+[0] = \mathbf{c}$ and $s_+[0] = \mathbf{s}$

Remark 22. *For every $n \in \mathbb{N}_+$, the sequential cardinal $\gamma_+[n]$ applied to $\Gamma_+[n]$ is given by the binomial coefficients:*

$$
\begin{aligned}
|\Gamma_+[n]| &= (|\Gamma_+[n]_{-1}|, |\Gamma_+[n]_0|, |\Gamma[n]_1|, \cdots, |\Gamma_+[[n]_n|, |\varnothing|, |\varnothing|, \cdots) \\
&= \left(\binom{n+1}{0}, \binom{n+1}{1}, \binom{n+1}{2}, \cdots, \binom{n+1}{n+1}, 0, 0, \cdots \right).
\end{aligned}
$$

5.2. The Monoidal Categories $((\mathbf{Sets}_{\text{fin}})^{\Gamma_+^{op}}, \boxplus, \Gamma_+[-1])$ and $(\mathbf{Z}^{\mathbb{N}_+^{op}}, \boxplus, \mathbf{1}_{-1})$

In this subsection, we consider $((\mathbf{Sets}_{\text{fin}})^{\Gamma_+^{op}}, \boxplus, \Gamma_+[-1])$, which is a monoidal subcategory of $((\mathbf{Sets})^{\Gamma_+^{op}}, \boxplus, \Gamma_+[-1])$. The sequential cardinal functor preserves the monoidal structure:

Theorem 4. *The sequential cardinal functor*

$$
|\cdot| : ((\mathbf{Sets}_{\text{fin}})^{\Gamma_+^{op}}, \boxplus, \Gamma_+[-1]) \to (\mathbf{Z}^{\mathbb{N}_+^{op}}, \boxplus, \mathbf{1}_{-1})
$$

is $(2,0)$-monoidal, that is, the following holds true:

$$
|X \boxplus Y| = |X| \boxplus |Y|, \quad |\Gamma_+[-1]| = \mathbf{1}_{-1}
$$

for all $X, Y \in (\mathbf{Sets}_{fin})^{\Gamma_+^{op}}$.

Proof. From the join definition, we have that

$$
(X \boxplus Y)_m = \sqcup_{p+q=m-1} X_p \times Y_q,
$$

where $-1 \leq p$ and $-1 \leq q$. Therefore, the equality $|X \boxplus Y| = |X| \boxplus |Y|$ follows taking into account Definition 11. Namely:

$$
\begin{aligned}
|X \boxplus Y|_m = |(X \boxplus Y)_m| &= |\sqcup_{p+q=m-1} X_p \times Y_q| \\
&= \sum_{p+q=m-1} |X_p \times Y_q| \\
&= \sum_{p+q=m-1} |X|_p \times |Y|_q \\
&= (|X| \boxplus |Y|)_m.
\end{aligned}
$$

$\square$

The cone functors, for semi-simplicial sets and for augmented sequences, are related through the sequential cardinal functor:

Proposition 10. *The following diagrams are commutative:*

$$
\begin{array}{ccc}
(\mathbf{Sets}_{fin})^{\Gamma_+^{op}} & \xrightarrow{Con_l} & (\mathbf{Sets}_{fin})^{\Gamma_+^{op}} \\
{\scriptstyle |\cdot|} \downarrow & & \downarrow {\scriptstyle |\cdot|} \\
\mathbf{Z}^{\mathbb{N}_+^{op}} & \xrightarrow{con} & \mathbf{Z}^{\mathbb{N}_+^{op}}
\end{array}
\qquad
\begin{array}{ccc}
(\mathbf{Sets}_{fin})^{\Gamma_+^{op}} & \xrightarrow{Con_r} & (\mathbf{Sets}_{fin})^{\Gamma_+^{op}} \\
{\scriptstyle |\cdot|} \downarrow & & \downarrow {\scriptstyle |\cdot|} \\
\mathbf{Z}^{\mathbb{N}_+^{op}} & \xrightarrow{con} & \mathbf{Z}^{\mathbb{N}_+^{op}}
\end{array}
$$

Moreover, as $\gamma_+[0] = |\Gamma_+[0]| = \mathbf{c}$, we have $\gamma_+[n] = |\Gamma_+[n]| = \boxplus^{n+1} \mathbf{c}$ (the $(n+1)$-fold join of $\mathbf{c}$ with itself).

Proof. By the definition of a left-cone functor, we have that

$$
|\mathrm{Con}_l(X)| = |\Gamma_+[0] \boxplus X| = |\Gamma_+[0]| \boxplus |X| = \mathbf{c} \boxplus |X| = \mathrm{con}(|X|).
$$

Here, we have used Theorem 4 and Proposition 7. The commutativity of the right diagram is analogously verified. The rest of the proof is straightforwardly taking into account the formula $\Gamma_+[n] \boxplus \Gamma_+[m] \cong \Gamma_+[n+m+1]$ and that $|X \boxplus Y| = |X| \boxplus |Y|$. $\square$

Definition 21. *Consider a finite semi-simplicial set* $X \in (\mathbf{Sets}^{\Gamma_+^{op}})_{fin}$ *and a rational number* $q \in \mathbb{Q}$*. Then, the Euler q-characteristic of X is the rational number defined for* $X \neq \varnothing$ *as*

$$E_q(X) = \sum_{i=-1}^{dim(X)} (-q)^i |X|_i$$

and $E_q(\varnothing) = 0$*.*

Corollary 8. *If* $X \in (\mathbf{Sets}^{\Gamma_+^{op}})_{fin}$ *is, up to isomorphism, the (left or right) cone of some* $Y \in (\mathbf{Sets}^{\Gamma_+^{op}})_{fin}$*, then* $E_1(X) = 0$ *and for* $k \in \mathbb{N}_+$ *such that* $-1 \leq k \leq dim(X)$*,* $\sum_{i=-1}^{k} (-1)^{k-i} |X|_i \geq 0$*.*

Proof. If $X \cong \mathrm{Con}_l(Y)$, then $|X| = |\mathrm{Con}_l(Y)| = \mathrm{con}(|Y|)$ and therefore $E_1(X) = E_1(|X|) = E_1(\mathrm{con}(|Y|)) = 0$ (see Remark 19). Note also that $|Y| = \mathrm{con}^{-1}(|X|)$ and $|Y|_k \geq 0$. By Remark 17, $|Y|_k = (\mathrm{con}^{-1}(|X|))_k = \sum_{i=-1}^{k} (-1)^{k-i} |X|_i \geq 0$. $\square$

Note that the cardinal functor $|\cdot| \colon \mathbf{Sets}_{fin} \to \mathbf{Z}$ induces a canonical functor $|\cdot| \colon ((\mathbf{Sets}_{fin})^{\Gamma_+^{op}})^{\Gamma_+} \to (\mathbf{Z}^{\mathbb{N}_+^{op}})^{\mathbb{N}}$, $Z \to |Z|$, where $|Z| = |\cdot| \circ Z \circ in$ is the composite

$$\mathbb{N}_+ \xrightarrow{\ in\ } \Gamma_+ \xrightarrow{\ Z\ } (\mathbf{Sets}_{fin})^{\Gamma_+^{op}} \xrightarrow{\ |\cdot|\ } \mathbf{Z}^{\mathbb{N}_+^{op}}.$$

Definition 22. *The augmented Pascal matrix is defined as the augmented co-sequence given by the composite:*

$$\mathbb{N}_+ \xrightarrow{\ in\ } \Gamma_+ \xrightarrow{\ \mathbf{Y}\ } (\mathbf{Sets}_{fin})^{\Gamma_+^{op}} \xrightarrow{\ |\cdot|\ } \mathbf{Z}^{\mathbb{N}_+^{op}}.$$

Note that $|\cdot| \circ \mathbf{Y} \circ in = \mathrm{bin}$ (given in Section 4.4).

Corollary 9. *The functor*

$$|\cdot| \colon ((\mathbf{Sets}_{fin})^{\Gamma_+^{op}})^{\Gamma_+} \to (\mathbf{Z}^{\mathbb{N}_+^{op}})^{\mathbb{N}}$$

carries the Yoneda augmented semi-cosimplicial set $\mathbf{Y} \colon \Gamma_+ \to \mathbf{Sets}^{\Gamma_+^{op}}$*, matricially represented as*

$$\mathbf{Y} = \begin{pmatrix} \Gamma_+[-1] \\ \Gamma_+[0] \\ \vdots \end{pmatrix}$$

to the augmented Pascal matrix, where each row is the cone of the previous one.
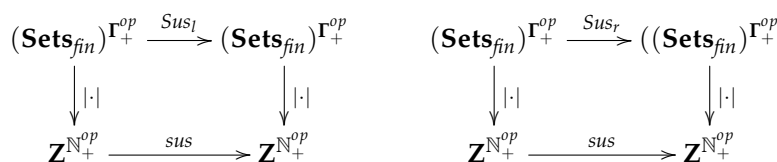
$$|\mathbf{Y}| = \begin{pmatrix} |\Gamma_+[-1]| \\ |\Gamma_+[0]| \\ \vdots \end{pmatrix} = \begin{pmatrix} \gamma_+[-1] \\ \gamma_+[0] \\ \vdots \end{pmatrix} = \mathrm{bin}$$

Remark 23. *Observe that iterating the cone construction to $|\Gamma_+[-1]| = \mathbf{1}_{-\mathbf{1}}$ generates, in a natural way, the well-known Pascal triangle:*

	-1	0	1	2	3	4	5	6	$\cdots$	
$\gamma_+[-1] = \|\Gamma_+[-1]\|$	1	0	0	0	0	0	0	0	$\cdots$	$\gamma_+[-1]$
$\gamma_+[0] = \|\Gamma_+[0]\|$	1	1	0	0	0	0	0	0	$\cdots$	$con\,\gamma_+[-1]$
$\gamma_+[1] = \|\Gamma_+[1]\|$	1	2	1	0	0	0	0	0	$\cdots$	$con^2\gamma_+[-1]$
$\gamma_+[2] = \|\Gamma_+[2]\|$	1	3	3	1	0	0	0	0	$\cdots$	$con^3\gamma_+[-1]$
$\gamma_+[3] = \|\Gamma_+[3]\|$	1	4	6	4	1	0	0	0	$\cdots$	$con^4\gamma_+[-1]$
$\gamma_+[4] = \|\Gamma_+[4]\|$	1	5	10	10	5	1	0	0	$\cdots$	$con^5\gamma_+[-1]$
$\gamma_+[5] = \|\Gamma_+[5]\|$	1	6	15	20	15	6	1	0	$\cdots$	$con^6\gamma_+[-1]$
$\gamma_+[6] = \|\Gamma_+[6]\|$	1	7	21	35	35	21	7	1	$\cdots$	$con^7\gamma_+[-1]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$

The suspension functors for semi-simplicial sets and for augmented sequences of integers are related as follows:

Proposition 11. *The following diagrams are commutative:*

$$
\begin{array}{ccc}
(\mathbf{Sets}_{fin})^{\Gamma_+^{op}} & \xrightarrow{\ Sus_l\ } & (\mathbf{Sets}_{fin})^{\Gamma_+^{op}} \\
\downarrow{\scriptstyle |\cdot|} & & \downarrow{\scriptstyle |\cdot|} \\
\mathbf{Z}^{\mathbb{N}_+^{op}} & \xrightarrow{\ sus\ } & \mathbf{Z}^{\mathbb{N}_+^{op}}
\end{array}
\qquad
\begin{array}{ccc}
(\mathbf{Sets}_{fin})^{\Gamma_+^{op}} & \xrightarrow{\ Sus_r\ } & ((\mathbf{Sets}_{fin})^{\Gamma_+^{op}} \\
\downarrow{\scriptstyle |\cdot|} & & \downarrow{\scriptstyle |\cdot|} \\
\mathbf{Z}^{\mathbb{N}_+^{op}} & \xrightarrow{\ sus\ } & \mathbf{Z}^{\mathbb{N}_+^{op}}
\end{array}
$$

Moreover, $|S_+[0]| = \mathbf{s} = s_+[0]$.

Below we have obtained, coming from $S^0 = S_+[0]$, the iteration of the suspension construction:

	-1	0	1	2	3	4	5	$\cdots$	
$\|S^0\|$	1	2	0	0	0	0	0	$\cdots$	$\mathbf{s}$
$\|Sus_l(S^0)\|$	1	4	4	0	0	0	0	$\cdots$	$sus(\mathbf{s})$
$\|Sus_l^2(S^0)\|$	1	6	12	8	0	0	0	$\cdots$	$sus^2(\mathbf{s})$
$\|Sus_l^3(S^0)\|$	1	8	24	32	16	0	0	$\cdots$	$sus^3(\mathbf{s})$
$\|Sus_l^4(S^0)\|$	1	10	40	80	80	32	0	$\cdots$	$sus^4(\mathbf{s})$
$\|Sus_l^5(S^0)\|$	1	12	60	160	240	192	64	$\cdots$	$sus^5(\mathbf{s})$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$

For each $n \in \mathbb{N}_+$, we may consider an operator $an_n \colon \mathbf{Z}^{\mathbb{N}_+^{op}} \to \mathbf{Z}^{\mathbb{N}_+^{op}}$, defined as

$$(an_n(c))_i = (1 - \delta_{n,i})c_i.$$

Note that $s_+[n] = an_{n+1}(\gamma_+[n+1])$. Therefore, one has:

	-1	0	1	2	3	4	5	$\cdots$	
$s_+[-1] = \|S_+[-1]\|$	1	0	0	0	0	0	0	$\cdots$	$an_0\gamma_+[0]$
$s_+[0] = \|S_+[0]\|$	1	2	0	0	0	0	0	$\cdots$	$an_1\gamma_+[1]$
$s_+[1] = \|S_+[1]\|$	1	3	3	0	0	0	0	$\cdots$	$an_2\gamma_+[2]$
$s_+[2] = \|S_+[2]\|$	1	4	6	4	0	0	0	$\cdots$	$an_3\gamma_+[3]$
$s_+[3] = \|S_+[3]\|$	1	5	10	10	5	0	0	$\cdots$	$an_4\gamma_+[4]$
$s_+[4] = \|S_+[4]\|$	1	6	15	20	15	6	0	$\cdots$	$an_5\gamma_+[5]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$

which is obtained by removing the first row and the principal diagonal in the augmented Pascal matrix.

For the suspension functor, the analogous of Corollary 8 is the following result:

Corollary 10. *Let X be a finite semi-simplicial set ($X \in (\mathbf{Sets}^{\Gamma_+^{op}})_{fin}$). If X is the suspension of some $Y \in (\mathbf{Sets}^{\Gamma_+^{op}})_{fin}$, then $E_{\frac{1}{2}}(X) = 0$ and for $k \in \mathbb{N}_+$ such that $-1 \leq k \leq dim(X)$, $\sum_{i=-1}^{k}(-2)^{k-i}|X|_i \geq 0$.*

5.3. The Categorical Semi-Rings $(\mathbf{Sets}^{\Gamma_+^{op}}, \sqcup, \boxplus, \Gamma_+^{\varnothing}, \Gamma_+[-1])$ and $(\mathbf{Z}^{\mathbb{N}_+^{op}}, +, \boxplus, 0, \mathbf{1}_{-1})$

As we have earlier commented, in this paper, by a *categorical semi-ring*, we will mean a symmetric bimonoidal category. This is just a category $\mathbf{C}$ equipped with two symmetric monoidal category structures:

- $(\mathbf{C}, \oplus, 0,)$ the *additive structure*
- $(\mathbf{C}, \otimes, I)$ the *multiplicative structure*

together with

(i) natural isomorphisms $0 \otimes A \xrightarrow{\cong} 0 \xleftarrow{\cong} A \otimes 0$, which are called *left multiplicative zero* and *right multiplicative zero*, respectively.

(ii) natural monomorphisms

$$A \otimes (B \oplus C) \longrightarrow (A \otimes B) \oplus (A \otimes C)$$

$$(A \oplus B) \otimes C \longrightarrow (A \otimes C) \oplus (B \otimes C)$$

called the *left distributivity morphism* and the *right distributivity morphism*, respectively.

This structure must satisfy Laplaza's axioms (see Definition 2.1.1 in [27] and [19,28]).

A functor $F : \mathbf{C} \to \mathbf{C}'$ between categorical semi-rings (or symmetric bimonoidal categories) is said to be a *functor of categorical semi-rings* if it is a symmetric bimonoidal functor, that is,

- F is a symmetric $(2,0)$-monoidal functor (recall from Definition 4) from the additive structure of $\mathbf{C}$ to the additive structure of $\mathbf{C}'$.
- F is a symmetric $(2,0)$-monoidal functor from the multiplicative structure of $\mathbf{C}$ to the multiplicative structure of $\mathbf{C}'$.

These are required to make the following two diagrams in $\mathbf{C}'$ commutative for all objects A, B, C:

Multiplicative zero:
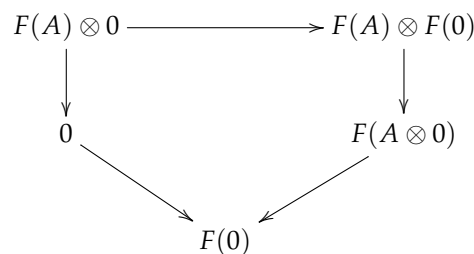
Distributivity:

$$
\begin{array}{ccc}
(F(A) \oplus F(B)) \otimes F(C) & \longrightarrow & (F(A) \otimes F(C)) \oplus (F(B) \otimes F(C)) \\
\downarrow & & \downarrow \\
F(A \oplus B) \otimes F(C) & & F(A \otimes C) \oplus F(B \otimes C) \\
\downarrow & & \downarrow \\
F((A \oplus B) \otimes C) & \longrightarrow & F((A \otimes C) \oplus (B \otimes C))
\end{array}
$$

The next result gives a functor of categorical semi-rings:

Proposition 12. *The sequential cardinal functor*

$$
|\cdot| : ((\mathbf{Sets}_{fin})^{\Gamma_+^{op}}, \sqcup, \boxplus, \Gamma_+^{\varnothing}, \Gamma_+[-1]) \to (\mathbf{Z}^{\mathbb{N}_+^{op}}, +, \boxplus, 0, \mathbf{1}_{-1})
$$

is a functor of categorical semi-rings such that the following diagram is commutative

$$
\begin{array}{ccc}
\mathbf{\Gamma}_+ & \xrightarrow{\gamma_+} & \mathbf{Z}^{\mathbb{N}_+^{op}} \\
{\scriptstyle \mathbf{Y}}\downarrow & \nearrow{\scriptstyle (-)\widetilde{\rhd}\gamma_+} & \\
\mathbf{Sets}^{\Gamma_+^{op}} & &
\end{array}
$$

Proof. Consider the monoidal category $(\mathbf{Z}^{\mathbb{N}_+^{op}}, \boxplus, \mathbf{1}_{-1})$. The finite sequences $\mathbf{1}_{-1} = \gamma_+[-1] = |\Gamma_+[-1]|$, $\mathbf{1}_{-1} + \mathbf{1}_0 = \gamma_+[0] = |\Gamma_+[0]|$ determine a unique morphism, $\gamma_+[-1] \to \gamma_+[0]$, in $\mathbf{Z}^{\mathbb{N}_+^{op}}$. Applying Proposition 1, we obtain a (symmetric) $(2,0)$-monoidal functor $(\mathbf{\Gamma}_+, \sqcup, [-1]) \to (\mathbf{Z}^{\mathbb{N}_+^{op}}, \boxplus, \mathbf{1}_{-1})$ given, up to isomorphism, by $[n] \mapsto \gamma_+[n]$

$$
\begin{array}{ccc}
\mathbf{\Gamma}_+|_{[-1],[0]} & \longrightarrow & \mathbf{Z}^{\mathbb{N}_+^{op}} \\
{\scriptstyle \mathrm{in}}\downarrow & \nearrow{\scriptstyle \gamma_+} & \\
\mathbf{\Gamma}_+ & &
\end{array}
$$

Taking into account Remark 5, one has a commutative diagram

$$
\begin{array}{ccc}
\mathbf{\Gamma}_+ & \xrightarrow{\gamma_+} & \mathbf{Z}^{\mathbb{N}_+^{op}} \\
{\scriptstyle \mathbf{Y}}\downarrow & \nearrow{\scriptstyle (-)\widetilde{\rhd}\gamma_+} & \\
\mathbf{Sets}^{\Gamma_+^{op}} & &
\end{array}
$$

where $(-)\widetilde{\rhd}\gamma_+ : ((\mathbf{Sets}_{fin})^{\Gamma_+^{op}}, \boxplus, \Gamma_+[-1]) \to (\mathbf{Z}^{\mathbb{N}_+^{op}}, \boxplus, \mathbf{1}_{-1})$ is a $(2,0)$-monoidal functor. Since, by Theorem 4, we also have the (symmetric) $(2,0)$-monoidal functor

$$
|\cdot| : ((\mathbf{Sets}_{fin})^{\Gamma_+^{op}}, \boxplus, \Gamma_+[-1]) \to (\mathbf{Z}^{\mathbb{N}_+^{op}}, \boxplus, \mathbf{1}_{-1})
$$

such that $|\mathbf{Y}([n])| = \gamma_+(n)$, it follows that $(-)\widetilde{\rhd}\gamma_+ = |\cdot|$. As it preserves coproducts it is, actually, a functor of categorical semi-rings:

$$
|\cdot| : ((\mathbf{Sets}_{fin})^{\Gamma_+^{op}}, \sqcup, \boxplus, \Gamma_+^{\varnothing}, \Gamma_+[-1]) \to (\mathbf{Z}^{\mathbb{N}_+^{op}}, +, \boxplus, 0, \mathbf{1}_{-1})
$$

We leave the rest of the proof to the reader. $\square$

5.4. Comparing Actions and Triangle Products

To finish this section, we establish an important result asserting that, under mild restrictions, the sequential cardinal functor carries the action to the triangle product. We need a previous definition.

Definition 23. *A functor* $Z : \Gamma_+ \to \mathbf{Sets}^{\Gamma_+^{op}}$ *is said to be* regular *if* $Z(\varphi)$ *is injective (on each dimension) for every morphism* φ *in* Γ_+. *We shall denote as* $(\mathbf{Sets}^{\Gamma_+^{op}})_{reg}^{\Gamma_+}$ *the full subcategory of* $(\mathbf{Sets}^{\Gamma_+^{op}})^{\Gamma_+}$ *consisting of regular functors.*

Theorem 5. *The following diagram is commutative:*

$$
\begin{array}{ccc}
(\mathbf{Sets}_{fin})^{\Gamma_+^{op}} \times ((\mathbf{Sets}_{fin})^{\Gamma_+^{op}})_{reg}^{\Gamma_+} & \xrightarrow{\ (-)\widetilde{\rhd}(-)\ } & (\mathbf{Sets}_{fin})^{\Gamma_+^{op}} \\
{\scriptstyle |-|\times|-|}\Big\downarrow & & \Big\downarrow{\scriptstyle |-|} \\
\mathbf{Z}^{\mathbb{N}_+^{op}} \times (\mathbf{Z}^{\mathbb{N}_+^{op}})^{\mathbb{N}_+} & \xrightarrow[\ (-)\widetilde{\rhd}(-)\ ]{} & \mathbf{Z}^{\mathbb{N}_+^{op}}
\end{array}
$$

In other words, if $X \in (\mathbf{Sets}_{fin})^{\Gamma_+^{op}}$ *and* $Z \in ((\mathbf{Sets}_{fin})^{\Gamma_+^{op}})_{reg}$, *then*

$$
|X\widetilde{\rhd}Z| = |X|\widetilde{\rhd}|Z|.
$$

Moreover, if we specialize $Z := \Gamma_+[-]$, *then* $|X\widetilde{\rhd}\Gamma_+[-]| = |X|\widetilde{\rhd}|\Gamma_+[-]| = |X|$.

Proof. If $Z : \Gamma_+ \to \mathbf{Sets}^{\Gamma_+^{op}}$ is regular, then we consider

$$
\begin{aligned}
\check{Z}([-1]) &:= Z([-1]) \\
\check{Z}([p]) &:= Z([p]) \setminus (\cup_{i=0}^p (\varphi_i)_*(Z([p-1]))), \quad p \geq 0
\end{aligned}
$$

From this construction, one can easily check that

$$
|Z([p])| = \sum_{k=-1}^{p} \binom{p+1}{k+1} |\check{Z}([k])|,
$$

where $|Z([p])|, |\check{Z}([k])| \in \mathbf{Z}^{\mathbb{N}_+^{op}}$. Therefore, we have the following product of matrices:

$$
|Z| = \mathrm{bin} \cdot |\check{Z}|.
$$

Finally, from the definition of $X\widetilde{\rhd}Z$, we obtain

$$
|X\widetilde{\rhd}Z| = |X| \cdot |\check{Z}| = |X| \cdot (\mathrm{bin}^{-1} \cdot |Z|) = |X|\widetilde{\rhd}|Z|.
$$

□

6. Example: Functors and Triangle-Products Induced by the Dunce Cap

To finish our study, we provide a striking result coming from the example of the dunce cap.

6.1. Augmented Triangular Numbers

Definition 24. *Let* $d \geq 0$ *be an integer. For* $n \geq 1$, *we consider* T_n^d, *the n-th triangular number of dimension* d, *inductively defined as*

$$
T_n^0 = 1, \quad T_n^d = \sum_{i=1}^{n} T_i^{d-1}, d \geq 1.
$$

We refer the reader to Theorem 1.3 in [13] for a proof of the following result ($d \geq 1$, $n \geq 1$):

$$T_n^d = \frac{1}{d!} n(n+1) \cdots (n+d-1).$$

We complete the matrix of triangular numbers for dimension -1 to obtain the augmented matrix $\mathrm{tri} \in (\mathbf{Z}^{\mathbb{N}_+^{op}})^{\mathbb{N}_+}$ of triangular numbers $\mathrm{tri}_{i,j}$, $i,j \in \mathbb{N}_+$

$$\mathrm{tri}_{i,j} = \begin{cases} \delta_{-1,j}, & \text{if } i = -1 \\ T_{j+2}^i, & \text{if } i \geq 0. \end{cases}$$

Note that, for $i \geq 1$, $j \geq -1$, one has that $\mathrm{tri}_{i,j} = T_{j+2}^i = \frac{(j+2)(j+3)\cdots(j+1+i)}{i!} = \binom{j+1+i}{i} = \mathrm{bin}_{i+j,i-1}$. Then, it is easy to check that, for $i,j \in \mathbb{N}_+$, $i \geq 0$, we also have:

$$\mathrm{tri}_{i,j} = \mathrm{bin}_{i+j,i-1}.$$

We give some values of the matrix tri:

tri_{ij}	-1	0	1	2	3	4	5	6	6	8	$\cdots$
-1	1	0	0	0	0	0	0	0	0	0	$\cdots$
0	1	1	1	1	1	1	1	1	1	1	$\cdots$
1	1	2	3	4	5	6	7	8	9	10	$\cdots$
2	1	3	6	10	15	21	28	36	45	55	$\cdots$
3	1	4	10	20	35	56	84	120	165	220	$\cdots$
4	1	5	15	35	70	126	210	330	495	715	$\cdots$
5	1	6	21	56	126	252	462	792	1287	2002	$\cdots$
6	1	7	28	84	210	462	924	1716	3003	5005	$\cdots$
7	1	8	36	120	330	792	1716	3432	6435	11400	$\cdots$
8	1	9	45	165	495	1287	3003	6435	12870	14400	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

6.2. Triangular Numbers and Co-Semi-Simplicial Object Induced by the Dunce Cap

Associated with the dunce cap Γ_+^1, which is the final object in the category $\mathbf{Sets}^{\Gamma_+^{op}}$, we can consider the unique augmented semi-simplicial map

$$f : \Gamma_+[-1] \to \Gamma_+^1.$$

Corollary 11. *Let $\widetilde{\Gamma}_+^1 : \Gamma_+|_{[-1],[0]} \to \mathbf{Sets}^{\Gamma_+^{op}}$ the functor induced by the map $f : \Gamma_+[-1] \to \Gamma_+^1$. Then, there exists an extension*

$$
\begin{array}{ccc}
\Gamma_+|_{[-1],[0]} & \xrightarrow{\widetilde{\Gamma}_+^1} & \mathbf{Sets}^{\Gamma_+^{op}} \\
{\scriptstyle in}\big\downarrow\ \ & \nearrow{\scriptstyle \boxplus\Gamma_+^1} & \\
\Gamma_+ & &
\end{array}
$$

given as

$$
\begin{aligned}
(\boxplus\Gamma_+^1)([-1]) &= \Gamma_+[-1], & n &= -1 \\
(\boxplus\Gamma_+^1)([n]) &= \boxplus_0^n \Gamma_+^1, & n &\geq 0 \\
(\boxplus\Gamma_+^1)([-1] \to [0]) &= f. &
\end{aligned}
$$

Moreover, $\boxplus\Gamma_+^1 : (\Gamma_+, \sqcup, [-1]) \to (\mathbf{Sets}^{\Gamma_+^{op}}, \boxplus, \Gamma_+[-1])$ is, up to isomorphism, the unique $(2,0)$-monoidal extension.

Remark 24. *If we consider* $(\boxplus\Gamma^1_+)([n]) = \boxplus^n_0\Gamma^1_+ \in (\mathbf{Sets}_{fin})^{\Gamma^{op}_+}$, *then we have that*

$$|(\boxplus\Gamma^1_+)([-1])| = (1,0,0,\cdots),$$

$$|(\boxplus\Gamma^1_+)([0])| = (1,1,1,\cdots).$$

Observe that $|(\boxplus\Gamma^1_+)([0])|_j = T^0_{j+2}$ *for* $j \in \mathbb{N}_+$,

$$|(\boxplus\Gamma^1_+)([1])| = (1,2,3,4,\cdots).$$

Then, we have that $|(\boxplus\Gamma^1_+)([1])|_j = T^1_{j+2}$ *for* $j \in \mathbb{N}_+$. *In general, for* $d \geq 0$, *we have:*

$$|(\boxplus\Gamma^1_+)([d])|_j = T^d_{j+2}, \quad j \in \mathbb{N}_+.$$

Therefore, for any $i, j \in \mathbb{N}_+$,

$$|(\boxplus\Gamma^1_+)([i])|_j = \mathrm{tri}_{i,j}.$$

6.3. Functors of Categorical Semi-Rings Induced by the Dunce Cup

We can construct functors of categorical semi-rings using triangular products:

Corollary 12. *The dunce cup* $\Gamma^1_+ \in (\mathbf{Sets}_{fin})^{\Gamma^{op}_+}$ *induces the functor of categorical semi-rings:*

$$(-)\rhd\Gamma^1_+ : ((\mathbf{Sets}_{fin})^{\Gamma^{op}_+}, \sqcup, \boxplus, \Gamma^{\varnothing}_+, \Gamma_+[-1]) \to ((\mathbf{Sets}_{fin})^{\Gamma^{op}_+}, \sqcup, \boxplus, \Gamma^{\varnothing}_+, \Gamma_+[-1])$$

such that, if $X \in (\mathbf{Sets}_{fin})^{\Gamma^{op}_+}$

$$X\rhd\Gamma^1_+ = X\widetilde{\rhd}(\boxplus\Gamma^1_+).$$

In particular, for every $X, Y \in (\mathbf{Sets}_{fin})^{\Gamma^{op}_+}$, *we have the canonical isomorphism:*

$$(X\boxplus Y)\rhd\Gamma^1_+ \cong (X\rhd\Gamma^1_+) \boxplus (Y\rhd\Gamma^1_+).$$

Proof. This is just a consequence of Corollaries 6 and 11. □

Remark 25. *Observe that we can compute the sequential cardinal* $|X\rhd(\boxplus\Gamma^1_+)|$ *by means of the matricial product*
$$|X\widetilde{\rhd}(\boxplus\Gamma^1_+)| = |X| \cdot bin^{-1} \cdot tri$$

Corollary 13. *The dunce cup* Γ^1_+ *induces the functor of categorical semi-rings:*

$$|(-)\rhd\Gamma^1_+| : ((\mathbf{Sets}_{fin})^{\mathbf{\Gamma^{op}_+}}, \sqcup, \boxplus, \Gamma^{\varnothing}_+, \Gamma_+[-1]) \to (\mathbf{Z}^{\mathbb{N}^{op}_+}, +, \boxplus, 0, \mathbf{1}_{-1}),$$

$$X \to |X\rhd\Gamma^1_+| = |X\widetilde{\rhd}(\boxplus\Gamma^1_+)| = |X| \cdot bin^{-1} \cdot tri$$

Proof. It suffices to apply Corollary 12, Proposition 12 and Theorem 5. □

Example 1. *For the augmented semi-simplicial sets* $X = S_+[1]$, $Y = S_+[2]$, *we have the cardinal augmented sequences* $|X| = (1,3,3,0,\cdots)$ *and* $|Y| = (1,4,6,4,0,\cdots)$. *Using the formula* $|X\boxplus Y| = |X|\boxplus|Y|$, *we obtain the sequence*

$$|X\boxplus Y| = (1,7,21,34,30,12,0,0,0,0,0,0,0,0,0,0\cdots)$$

Now, using the formula $|Z \tilde{\rhd}(\boxplus\Gamma^1_+)| = (|Z| \cdot \mathrm{bin}^{-1}) \cdot \mathrm{tri}$, one has the sequences

$$|X\tilde{\rhd}(\boxplus\Gamma^1_+)| = (1, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, \cdots)$$
$$|Y\tilde{\rhd}(\boxplus\Gamma^1_+)| = (1, 4, 10, 20, 34, 52, 74, 100, 130, 164, 202, 244, 290, 340, 394, \cdots)$$
$$|(X \boxplus Y)\tilde{\rhd}(\boxplus\Gamma^1_+)| = (1, 7, 28, 83, 202, 427, 812, 1423, 2338, 3647, 5452, 7867, \cdots)$$

On another note, one can check that computing the join of augmented sequences $|X\tilde{\rhd}(\boxplus\Gamma^1_+)|$, $|Y\tilde{\rhd}(\boxplus\Gamma^1_+)|$, we also have:

$$|X\tilde{\rhd}(\boxplus\Gamma^1_+)| \boxplus |Y\tilde{\rhd}(\boxplus\Gamma^1_+)| = (1, 7, 28, 83, 202, 427, 812, 1423, 2338, 3647, 5452, 7867, \cdots)$$

We point out that $|Y\tilde{\rhd}(\boxplus\Gamma^1_+)| = A005893$, where $A005893$ is the notation used in the encyclopedia of integer numbers (see [16,39]) to denote the number of points on the surface of a tetrahedron. However, the integer sequence $|(X \boxplus Y)\tilde{\rhd}(\boxplus\Gamma^1_+)|$ does not appear in this encyclopedia.

7. Conclusions and Future Work

Our main conclusion is that there is a nice relationship between augmented semi-simplicial finite sets and augmented sequences of integers. The categorical structure of semi-simplicial sets can be enriched with operations induced by finite coproduts, finite products, joins, and actions of some co-semi-simplicial objects and the space of augmented sequences admits some ring structures, and augmented matrices of integers can be used to transform augmented sequences. The sequential cardinal functor can be considered as an algebraization of the geometric objects given by augmented semi-simplicial finite sets into algebraic structures associated with augmented sequences.

In this paper, we have seen how the binomial numbers are connected with the Yoneda embedding and the iteration of the join operation on the final object of the category of semi-simplicial finite sets are connected with triangular numbers.

The authors have a project consisting of analyzing more properties of this algebraization process; in particular, we can see how the subdivision and the cylinder construction for semi-simplicial sets can be obtained by taking some actions of adequate co-semi-simplicial objects and how the sequential cardinal of a subdivision can be computed using chain-power numbers and Stirling numbers (see [14,40]).

A different interesting objective will be to construct a categorical semi-ring (or symmetric bimonoidal category structure)

$$((\mathbf{Sets}_{\mathrm{fin}})^{\Gamma^{op}_+}, \boxplus, \odot, \Gamma_+[-1], \Gamma_+[0])$$

verifying the following properties:

If $X, Y \in (\mathbf{Sets}_{\mathrm{fin}})^{\Gamma^{op}_+}$ and $\dim(X), \dim(Y)$ are finite, then

$$\dim(X \odot Y) = (\dim(X) + 1)(\dim(Y) + 1) - 1.$$

If $X, Y, Z \in (\mathbf{Sets}_{\mathrm{fin}})^{\Gamma^{op}_+}$, then

$$(X \boxplus Y) \odot Z \cong (X \odot Z) \boxplus (Y \odot Z).$$

Author Contributions: Investigation, J.M.G.-C., L.J.H.-P. and M.T.R.-R. All authors have contributed equally in this research paper. All authors have read and agreed to the published version of the manuscript.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Eilenberg, S.; Zilber, J.A. Semi-Simplicial Complexes and Singular Homology. *Ann. Math.* **1950**, *51*, 499–513. [CrossRef]
2. Buoncristiano, C.R.S.; Sanderson, B.J. *A Geometric Approach to Homology Theory*; London Mathematical Society Lecture Note Series; Cambridge University Press: Cambridge, UK, 1976; Volume 18.
3. Friedman, G. Survey Article: An elementary illustrated introduction to simplicial sets. *Rocky Mt. J. Math.* **2012**, *42*, 353–423. [CrossRef]
4. Verity, D. Weak complicial sets, a simplicial weak omega-category theory. Part I: Basic homotopy theory. *arXiv* **2006**, arXiv:math/0604414v3.
5. CooperativeWork. Collaborative nLab: Semi-Simplicial Object. Available online: https://ncatlab.org/nlab/history/semi-simplicial (accessed on 23 December 2021).
6. Ebert, J.; Randal-Williams, O. Semisimplicial spaces. *Algebr. Geom. Topol.* **2019**, *19*, 2099–2150. [CrossRef]
7. Gabriel, M.Z.P. *Calculus of Fractions and Homotopy Theory*; Springer: Berlin/Heidelberg, Germany, 1995.
8. Graham, R.L.; Knuth, D.E.; Patashnik, O. *Concrete Mathematics*; Addison-Wesley: Boston, MA, USA, 1990.
9. Greene, D.H.; Knuth, D.E. *Mathematics for the Analysis of Algorithms*, 3rd ed.; Birkhäuser: Boston, MA, USA, 1990.
10. Donaghey, R. Binomial self-inverse sequences and tangent coefficients. *J. Comb. Theory Ser. A* **1976**, *21*, 155–163. [CrossRef]
11. Donaghey, R.; Shapiro, L.W. Motzkin numbers. *J. Comb. Theory Ser. A* **1977**, *23*, 291–301. [CrossRef]
12. Riordan, J. *Combinatorial Identities*; Wiley: New York, NY, USA, 1968.
13. Kim, H.K. On Regular Polytope Numbers. *Proc. Am. Math. Soc.* **2003**, *131*, 65–75. [CrossRef]
14. Comtet, L. *Advanced Combinatorics: The Art of Finite and Infinite Expansion*; D. Reidel Pub. Co.: Dordrecht, The Netherlands; Boston, MA, USA, 1974.
15. Nelsen, R.B.; Schmidt, H. Chains in Power Sets. *Math. Mag.* **1991**, *64*, 23–31. [CrossRef]
16. Sloane, N.J.A.; Plouffe, S. *The Encyclopedia of Integer Sequences*; Academic Press: Cambridge, MA, USA, 1995.
17. Biggs, N.L. *Discrete Mathematics*; Clarendon Press: Oxford, UK, 1985.
18. Ehlers, P.; Porter, T. Joins for (augmented) simplicial sets. *J. Pure Appl. Algebra* **2000**, *145*, 37–44. [CrossRef]
19. Johnson, N.; Yau, D. Bimonoidal Categories, E_n-Monoidal Categories, and Algebraic K-Theory. *arXiv* **2021**, arXiv:2107.10526 .
20. Elgueta, J. The groupoid of finite sets is biinitial in the 2-category of rig categories. *J. Pure Appl. Algebra* **2021**, *225*, 106738. [CrossRef]
21. Yau, D. *2Dimensional Categories*; Oxford University Press: Oxford, UK, 2021.
22. Björner, A. Face Number of Complexes and Polytopes. In Proceedings of the International Congress Mathematicians, Berkeley, CA, USA, 3–11 August 1986.
23. Codenotti, G.; Katthän, L.; Sanyal, R. On f- and h-vectors of relative simplicial complexes. *Algebr. Comb.* **2019**, *2*, 343–353. [CrossRef]
24. Athanasiadis, C.A. Face numbers of uniform triangulations of simplicial complexes. *arXiv* **2021**, arXiv:2003.13372.
25. Nevo, E. Complexity yardsticks for f-vectors of polytopes and spheres. *arXiv* **2019**, arXiv:1908.09628.
26. Lane, S.M.; Moerdijk, I. *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*; Springer: New York, NY, USA, 1992.
27. Laplaza, M. Coherence for distributivity. In *Coherence in Categories*; Lecture Notes in Mathematics; Kelly, G., Laplaza, M., Lewis, G., Lane, S.M., Eds.; Springer: Berlin/Heidelberg, Germany, 1972; Volume 281, pp. 29–65.
28. Laplaza, M. A new result of coherence for distributivity. In *Coherence in Categories*; Lecture Notes in Mathematics; Kelly, G., Laplaza, M., Lewis, G., Lane, S.M., Eds.; Springer: Berlin/Heidelberg, Germany, 1972; Volume 281, pp. 214–235.
29. Joyal, A.; Street, R. Braided Tensor Categories. *Adv. Math.* **1993**, *102*, 20–78. [CrossRef]
30. MacLane, S. *Categories for the Working Mathematician*; Graduate Texts in Mathematics; Springer: New York, NY, USA, 1971; Volume 5; pp. i–ix, 1–262 .
31. Kelly, G.M. Basic Concepts of Enriched Category Theory. *Repr. Theory Appl. Categ.* **2005**, *14*, 1–18.
32. Carrasco, P.; Cegarra, A.; Garzón, A. Classifying spaces for braided monoidal categories and lax diagrams of bicategories. *Adv. Math.* **2021**, *226*, 419–483. [CrossRef]
33. Cegarra, A.M. Nonabelian cocycles and the spectrum of a symmetric monoidal category. *Turk. J. Math.* **2020**, *44*, 50–84. [CrossRef]
34. Kong, L.; Yuan, W.; Zhang, Z.H.; Zheng, H. Enriched monoidal categories I: Centers. *arXiv* **2021**, arXiv:2104.03121.
35. Dell, Z. A Characterization of Braided Enriched Monoidal Categories. *arXiv* **2021**, arXiv:2104.07747.
36. Trujillo, L. A Coherent Proof of Mac Lane's Coherence Theorem. Bachelor's Thesis, Harvey Mudd College, Claremont, CA, USA , 2020.
37. CooperativeWork. Collaborative Wiki nLab: Day Convolution. 2020. Available online: https://ncatlab.org/nlab/show/Day+convolution (accessed on 23 December 2021).
38. Milewski, B. Free Monoidal Functors. 2018. Available online: https://bartoszmilewski.com/2018/05/16/free-monoidal-functors-categorically/ (accessed on 23 December 2021).

39. CooperativeWork. The Online Encyclopedia of Integer Sequences. 2021. Available online: https://oeis.org (accessed on 23 December 2021).
40. Schader, M. Hierarchical Analysis: Classification with Ordinal Object Dissimilarities. *Metrika* **1980**, *27*, 127–132. [CrossRef]