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On Briançon–Skoda theorem for foliations[☆]

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Abstract

We generalize Mattei's result relative to the Briançon–Skoda theorem for foliations to the family of foliations of the second type. We use this generalization to establish relationships between the Milnor and Tjurina numbers of foliations of second type, inspired by the results obtained by Liu for complex hypersurfaces and we determine a lower bound for the global Tjurina number of an algebraic curve.

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1. Introduction and Statement of the results

The problem of deciding whether an element of a ring belongs to a given ideal of the ring is known as the *ideal membership* and dates back to works of Dedekind who gave the precised definition of an ideal. Even if we know generators of the ideal, it is not trivial to determine if an element is a member of it. Therefore it is interesting to give sufficient conditions for ideal membership. An important theorem in this line is the Hilbert's Nullstellensatz: it states that if I in an ideal in the ring of germs of holomorphic functions at $0 \in \mathbb{C}^n$ and f vanishes on the zero locus of I then there is a power of f belonging to I. The *Briançon–Skoda Theorem* can be seen as an *effective* version of the Hilbert Nullstellensatz when I is a jacobian ideal. Let us clarify this last statement. Let $f(x_1,\ldots,x_n) \in \mathbb{C}\{x_1,\ldots,x_n\}$ be a non-unit convergent power series. Consider its jacobian ideal $J(f) = (\partial_{x_1} f,\ldots,\partial_{x_n} f)$. According to Wall [18] it was Mather who asked about the smallest r for which $f^r \in J(f)$. It was known then that f is an element of the integral closure of $J(f)$, which implies the existence of a power of f belonging to $J(f)$. At that time it was also known, thanks to Saito [14], that if the origin is an isolated critical point of f then f belongs to $J(f)$ iff f is a quasi-homogeneous polynomial. Briançon and Skoda [16] proved, using analytic results of Skoda, that $f^n \in J(f)$. Later, Lipman and Teissier [8] gave an algebraic proof of this algebraic statement. Subsequently, Briançon–Skoda Theorem has been generalized in different contexts, and has given rise to abundant literature. In Foliation Theory, Mattei proved

Theorem 1. ([12, Théorème C]). *Let $\mathcal{F}$ be a non-dicritical generalized curve holomorphic foliation at $(\mathbb{C}^2, p)$ given by $\omega = P(x,y)dx + Q(x,y)dy$. If $f(x,y) = 0$ is the reduced curve of total union of separatrices of $\mathcal{F}$ then f^2 belongs to the ideal (P, Q).*

In this paper, we extend Theorem 1 to the family of second type foliations (perhaps dicritical) and show (see Example 3.2) that it is essential that the foliation be of the second type.

Theorem A. *Let $\mathcal{F}$ be a germ of a second type holomorphic foliation at $(\mathbb{C}^2, p)$ induced by $\omega = P(x,y)dx + Q(x,y)dy$, where $P, Q \in \mathbb{C}\{x,y\}$, and let $F = f/h$ be a reduced balanced equation of separatrices for $\mathcal{F}$. Then f^2 belongs to the ideal (P, Q).*

In Section 2 we introduce all the notions and tools necessary to prove Theorem A. We are inspired by Mattei's proof but to extend it to the dicritical case we use the characterizations of the dicritical second type foliations given by Genzmer in [6]. The proof of Theorem A is given in Section 3. In Section 4, we obtain relationships between the Milnor number, $\mu_p(\mathcal{F})$, and the Tjurina number, $\tau_p(\mathcal{F}, \mathcal{B}_0)$, of the foliation $\mathcal{F}$ with respect to the zero divisor $\mathcal{B}_0$ of a balanced divisor of separatrices $\mathcal{B} = \mathcal{B}_0 - \mathcal{B}_\infty$ of $\mathcal{F}$, inspiring us to do so in the work of Liu [9] for complex hypersurfaces. More precisely, if $\mathcal{P}^{\mathcal{F}}$ is a generic polar curve of $\mathcal{F}$, $\nu_p(.)$ denotes the algebraic multiplicity of a curve and $i_p(.,.)$ denotes the intersection multiplicity of two curves then we get

Theorem B. *Let $\mathcal{F}$ be a singular holomorphic foliation of second type at $(\mathbb{C}^2, p)$. Let $\mathcal{B} = \mathcal{B}_0 - \mathcal{B}_\infty$ be a balanced divisor of separatrices for $\mathcal{F}$. Then*

$$\frac{(\nu_p(\mathcal{B}_0)-1)^2 + \nu_p(\mathcal{B}_\infty) - i_p(\mathcal{P}^{\mathcal{F}}, \mathcal{B}_\infty) - i_p(\mathcal{B}_0, \mathcal{B}_\infty)}{2} \overset{(*)}{\leq} \frac{\mu_p(\mathcal{F})}{2} \leq \tau_p(\mathcal{F}, \mathcal{B}_0), \quad (1)$$

and the equality $(*)$ *holds if* $\mathcal{F}$ *is a generalized curve foliation and* $\mathcal{B}_0$ *is defined by a germ of semi-homogeneous function at* p*. Moreover, if* $\mathcal{B}_\infty = \emptyset$*, then*

$$\frac{\nu_p(\mathcal{F})^2}{2} \leq \frac{\mu_p(\mathcal{F})}{2} \leq \tau_p(\mathcal{F}, \mathcal{B}_0).$$

Finally, as consequence of Theorem B, in Section 5, we obtain a lower bound for the global Tjurina number of an algebraic curve.

2. Preliminaries

Let $\mathcal{F}$ be a germ of singular holomorphic foliation at $(\mathbb{C}^2, p)$, in local coordinates (x, y) centered at p, the foliation is given by a holomorphic 1-form

$$\omega = P(x, y)dx + Q(x, y)dy, \tag{2}$$

or by its dual vector field

$$v = -Q(x, y)\frac{\partial}{\partial x} + P(x, y)\frac{\partial}{\partial y}, \tag{3}$$

where $P(x, y), Q(x, y) \in \mathbb{C}\{x, y\}$ are relatively prime, and $\mathbb{C}\{x, y\}$ is the ring of complex convergent power series in two variables. The *algebraic multiplicity* of $\mathcal{F}$, denoted by $\nu_p(\mathcal{F})$, is the minimum of the orders $\nu_p(P)$, $\nu_p(Q)$ at p of the coefficients of ω.

We say that $C : f(x, y) = 0$, with $f(x, y) \in \mathbb{C}\{x, y\}$, is an $\mathcal{F}$*-invariant* curve if

$$\omega \wedge df = (f.h)dx \wedge dy,$$

for some $h \in \mathbb{C}\{x, y\}$. A *separatrix* of $\mathcal{F}$ is an irreducible $\mathcal{F}$-invariant curve. Denote by $Sep_p(\mathcal{F})$ the set of all separatrices of $\mathcal{F}$ through p. If $Sep_p(\mathcal{F})$ is a finite set then we say that the foliation $\mathcal{F}$ is *non-dicritical* and we call *total union of separatrices* of $\mathcal{F}$ to the union of all elements of $Sep_p(\mathcal{F})$. Otherwise we will say that $\mathcal{F}$ is a *dicritical* foliation.

A point $p \in \mathbb{C}^2$ is a *reduced* or *simple* singularity for $\mathcal{F}$ if the linear part $\mathrm{D}v(p)$ of the vector field v in (3) is non-zero and has eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$ fitting in one of the two following cases:

(i) $\lambda_1\lambda_2 \neq 0$ and $\lambda_1/\lambda_2 \notin \mathbb{Q}^+$ (in which case we say that p is a *non-degenerate* or *complex hyperbolic* singularity).
(ii) $\lambda_1 \neq 0$ and $\lambda_2 = 0$ (in which case we say that p is a *saddle–node* singularity).

The reduction process of the singularities of a codimension one singular foliation over an ambient space of dimension two was achieved by Seidenberg [15].

A singular foliation $\mathcal{F}$ at $(\mathbb{C}^2, p)$ is a *generalized curve foliation* if it has no saddle–nodes in its reduction process of singularities. This concept was defined by Camacho–Lins Neto–Sad [3, Page 144]. In this case, there is a system of coordinates (x, y) in which $\mathcal{F}$ is induced by the equation

$$\omega = x(\lambda_1 + a(x, y))dy - y(\lambda_2 + b(x, y))dx, \tag{4}$$

where $a(x, y), b(x, y) \in \mathbb{C}\{x, y\}$ are non-units, so that $Sep_p(\mathcal{F})$ is formed by two transversal analytic branches given by $\{x = 0\}$ and $\{y = 0\}$. In the case (2), up to a formal change of coordinates, the saddle–node singularity is given by a 1-form of the type

$$\omega = x^{k+1}dy - y(1+\lambda x^k)dx, \tag{5}$$

where $\lambda \in \mathbb{C}$ and $k \in \mathbb{Z}_{>0}$ are invariants after formal changes of coordinates (see [10, Proposition 4.3]). The curve $\{x = 0\}$ is an analytic separatrix, called *strong*, whereas $\{y = 0\}$ corresponds to a possibly formal separatrix, called *weak* or *central*.

Let $\mathcal{F}$ be a foliation at $(\mathbb{C}^2, p)$, given by a 1-form as in (2), with reduction process $\pi : (\tilde{X}, \mathcal{D}) \to (\mathbb{C}^2, p)$ and let $\tilde{\mathcal{F}} = \pi^*\mathcal{F}$ be the strict transform of $\mathcal{F}$. Denote by $\mathrm{Sing}(\cdot)$ the set of singularities of a foliation. A saddle–node singularity $q \in \mathrm{Sing}(\tilde{\mathcal{F}})$ is said to be a *tangent saddle–node* if its weak separatrix is contained in the exceptional divisor $\mathcal{D}$, that is, the weak separatrix is an irreducible component of $\mathcal{D}$.

A foliation is *in the second class* or is *of second type* if there are no tangent saddle–nodes in its reduction process of singularities. This notion was studied by Mattei–Salem [13] in the non-dicritical case and by Genzmer [6] for arbitrary foliations.

For a fixed reduction process of singularities $\pi : (\tilde{X}, \mathcal{D}) \to (\mathbb{C}^2, p)$ for $\mathcal{F}$, a component $D \subset \mathcal{D}$ can be:

- *non-dicritical*, if D is $\tilde{\mathcal{F}}$-invariant. In this case, D contains a finite number of simple singularities. Each non-corner singularity of D carries a separatrix transversal to D, whose projection by π is a curve in $Sep_p(\mathcal{F})$. Remember that a corner singularity of D is an intersection point of D with other irreducible component of $\mathcal{D}$.
- *dicritical*, if D is not $\tilde{\mathcal{F}}$-invariant. The reduction process of singularities gives that D may intersect only non-dicritical components of $\mathcal{D}$ and $\tilde{\mathcal{F}}$ is everywhere transverse to D. The π-image of a local leaf of $\tilde{\mathcal{F}}$ at each non-corner point of D belongs to $Sep_p(\mathcal{F})$.

Denote by $Sep_p(D) \subset Sep_p(\mathcal{F})$ the set of separatrices whose strict transforms by π intersect the component $D \subset \mathcal{D}$. If $B \in Sep_p(D)$ with D non-dicritical, B is said to be *isolated*. Otherwise, it is said to be a *dicritical separatrix*. This determines the decomposition $Sep_p(\mathcal{F}) = Iso_p(\mathcal{F}) \cup Dic_p(\mathcal{F})$, where notations are self-evident. The set $Iso_p(\mathcal{F})$ is finite and contains all purely formal separatrices. It subdivides further in two classes: *weak* separatrices – those arising from the weak separatrices of saddle–nodes – and *strong* separatrices – corresponding to strong separatrices of saddle–nodes and separatrices of non-degenerate singularities. On the other hand, if $Dic_p(\mathcal{F})$ is non-empty then it is an infinite set of analytic separatrices. Observe that a foliation $\mathcal{F}$ is *dicritical* when $Sep_p(\mathcal{F})$ is infinite, which is equivalent to saying that $Dic_p(\mathcal{F})$ is non-empty. Otherwise, $\mathcal{F}$ is *non-dicritical*.

Throughout the text, we would rather adopt the language of *divisors* of formal curves. More specifically, a *divisor of separatrices* for a foliation $\mathcal{F}$ at $(\mathbb{C}^2, p)$ is a formal sum

$$\mathcal{B} = \sum_{B \in \mathrm{Sep}_p(\mathcal{F})} a_B \cdot B, \tag{6}$$

where the coefficients $a_B \in \mathbb{Z}$ are zero except for finitely many $B \in Sep_p(\mathcal{F})$. The set of separatrices $\{B : a_B \neq 0\}$ appearing in (6) is called the *support* of the divisor $\mathcal{B}$ and it is denoted by $\mathrm{supp}(\mathcal{B})$. The *degree* of the divisor $\mathcal{B}$ is by definition $\deg \mathcal{B} = \sum_{B \in \mathrm{supp}(\mathcal{B})} a_B$.

We denote by $Div_p(\mathcal{F})$ the set of all these divisors of separatrices, which turns into a group with the canonical additive structure. We follow the usual terminology and notation:

- $\mathcal{B} \geq 0$ denotes an *effective* divisor, one whose coefficients are all non-negative;
- there is a unique decomposition $\mathcal{B} = \mathcal{B}_0 - \mathcal{B}_\infty$, where $\mathcal{B}_0, \mathcal{B}_\infty \geq 0$ are respectively the *zero* and *pole* divisors of $\mathcal{B}$;
- the *algebraic multiplicity* of $\mathcal{B}$ is $\nu_p(\mathcal{B}) = \sum_{B \in \mathrm{supp}(\mathcal{B})} \nu_p(B)$.

Following [6, page 5] and [7, Definition 3.1], we define a *balanced divisor of separatrices* for $\mathcal{F}$ as a divisor of the form

$$\mathcal{B} = \sum_{B \in \mathrm{Iso}_p(\mathcal{F})} B + \sum_{B \in \mathrm{Dic}_p(\mathcal{F})} a_B \cdot B,$$

where the coefficients $a_B \in \mathbb{Z}$ are non-zero except for finitely many $B \in Dic_p(\mathcal{F})$, and, for each dicritical component $D \subset \mathcal{D}$, the following equality is respected:

$$\sum_{B \in \mathrm{Dic}_p(D)} a_B = 2 - val(D).$$

The integer $val(D)$ stands for the *valence* of a component $D \subset \mathcal{D}$ in the reduction process of singularities, that is, it is the number of components of $\mathcal{D}$ intersecting D other from D itself.

The notion of balanced divisor of separatrices generalizes, to dicritical foliations, the notion of total union of separatrices for non-dicritical foliations.

A balanced divisor $\mathcal{B} = \sum_B a_B B$ of separatrices of $\mathcal{F}$ is called *primitive* if, $a_B \in \{-1, 1\}$ for any $B \in \mathrm{supp}(\mathcal{B})$. A *balanced equation of separatrices* is a formal meromorphic function $F(x, y)$ whose associated divisor $\mathcal{B} = \mathcal{B}_0 - \mathcal{B}_\infty$ is a balanced divisor. A balanced equation is *reduced* or *primitive* if the same is true for the underlying divisor.

By [6, Proposition 2.4] we have

$$\nu_p(\mathcal{F}) = \nu_p(\mathcal{B}) - 1 + \xi_p(\mathcal{F}) \tag{7}$$

and

$$\mathcal{F} \text{ is a second type foliation if, and only if, } \nu_p(\mathcal{F}) = \nu_p(\mathcal{B}) - 1, \tag{8}$$

where $\mathcal{B}$ is a balanced divisor of separatrices for $\mathcal{F}$ and $\xi_p(\mathcal{F})$ is the tangency excess of $\mathcal{F}$ at p (see [5, Definition 2.3]).

3. Proof of Theorem A

Let $\mathcal{F}$ be a germ of a singular holomorphic foliation at $(\mathbb{C}^2, p)$ induced by $\omega := P(x, y)dx + Q(x, y)dy$, where $P, Q \in \mathbb{C}\{x, y\}$, and consider $\pi := \sigma_1 \circ \ldots \circ \sigma_\ell : (\tilde{X}, \mathcal{D}) \to (\mathbb{C}^2, p)$ a reduction of singularities of $\mathcal{F}$ at $p \in \mathbb{C}^2$. Denote by $F = f/h$ a reduced balanced equation of separatrices for $\mathcal{F}$, and by Z_0 and Z_∞ the respective strict transforms by π of the curves $\{f = 0\}$ and $\{h = 0\}$. Let $\tilde{\mathcal{F}} = \pi^*(\mathcal{F})$ be the strict transform of $\mathcal{F}$ by π, $\mathcal{X}_{\tilde{\mathcal{F}}}$ be the sheaf of (holomorphic) vector fields tangent to $\tilde{\mathcal{F}}$, $\mathcal{X}_{Z_0}$ be the sheaf of vector fields tangent to the divisor $\mathcal{D}$ and to Z_0, and $H^1(\tilde{D}, \mathcal{X}_{\tilde{\mathcal{F}}})$ the first cohomology group of $\mathcal{X}_{\tilde{\mathcal{F}}}$ on $\tilde{D}$.

We have the following proposition:

Proposition 3.1. *Let $\mathcal{F}$ be a germ of a second type holomorphic foliation at $(\mathbb{C}^2, p)$ and let $F = f/h$ be a reduced balanced equation of separatrices for $\mathcal{F}$. Put $\varphi = f \circ \pi$. Then, the morphism*

$$[\varphi]\cdot : H^1(\mathcal{D}, \mathcal{X}_{\tilde{\mathcal{F}}}) \to H^1(\mathcal{D}, \mathcal{X}_{\tilde{\mathcal{F}}}), \quad [Y_{ij}] \longmapsto [\varphi \cdot Y_{ij}]$$

is identically zero.

Proof. We will prove by induction on the number ℓ of blow-ups needed to obtain the reduction of singularities of $\mathcal{F}$. If $\ell = 1$, then $H^1(\tilde{D}, \mathcal{X}_{\tilde{\mathcal{F}}})$ is of finite dimension by [11, Lemme 2.2.1]. From the proof of [11, Lemme 2.2.1] we have $X_{ij} = x_1^i y_1^j X_1$ such that $(i, j) \in I = \{(i, j) \in \mathbb{N} \times \mathbb{Z} : i \geq 0, \text{ and } i - \nu_p(\mathcal{F}) + \epsilon_p(\mathcal{F}) < j < 0\}$ induce a basis for $H^1(\tilde{D}, \mathcal{X}_{\tilde{\mathcal{F}}})$, where $x_1 = x$, $y_1 = \frac{y}{x}$ and $x_2 = \frac{x}{y}$, $y_2 = y$ are the local coordinates of the blow-up $\tilde{X}$, $X_1 = \frac{1}{x_1^{\nu_p(\mathcal{F})-\epsilon_p(\mathcal{F})}} \cdot v^{(1)}$, and $v^{(1)}$ is the vector field inducing $\sigma_1^*(\mathcal{F})$. Therefore the sections of the form

$$x_1^\alpha y_1^\beta X_1 \quad \text{such that} \quad (\alpha, \beta) \in \mathbb{N} \times \mathbb{Z}, \quad \beta \geq 0 \quad \text{or} \quad \beta \leq \alpha - \nu_p(\mathcal{F}) + \epsilon_p(\mathcal{F}) \tag{9}$$

are elements of $\check{B}(\mathcal{D}, \mathcal{X}_{\tilde{\mathcal{F}}})$ (i.e. 1-coboundary of $\mathcal{X}_{\tilde{\mathcal{F}}}$). Since $\mathcal{F}$ is of second type, $\nu_p(\mathcal{F}) = \nu_p(F) - 1$, which implies that $\nu_p(f) = \nu_p(h) + \nu_p(\mathcal{F}) + 1$. In particular, $\varphi = f \circ \pi \in (x_1^{\nu_p(h)+\nu_p(\mathcal{F})+1})$. Hence the sections $\varphi \cdot X_{ij}$ with $(i, j) \in I$ are elements of $\check{B}(\mathcal{D}, \mathcal{X}_{\tilde{\mathcal{F}}})$ and the proof of proposition ends for $\ell = 1$.

Now, for the general case, we use the exact sequence (see [11, page 312]):

$$0 \longrightarrow H^1(\tilde{D}, \mathcal{X}_{\tilde{\mathcal{F}}^1}) \xrightarrow{\rho} H^1(\mathcal{D}, \mathcal{X}_{\tilde{\mathcal{F}}}) \xrightarrow{\psi} H^1(D', \mathcal{X}_{\tilde{\mathcal{F}}}) \longrightarrow 0$$

where $\tilde{D} = \sigma_1^{-1}(p)$, $\tilde{\mathcal{F}}^1$ is the strict transform of $\mathcal{F}$ by σ_1, D' is the union of irreducible components of $\mathcal{D}$ different of $\tilde{D}$, ψ is the restriction morphism and ρ is the morphism induced by the natural inclusion of $\tilde{D}$ in $\mathcal{D}$. Finally, since the following diagram is commutative

$$\begin{tikzcd}
0 \arrow[r] & H^1(\tilde{D}, \mathcal{X}_{\tilde{\mathcal{F}}^1}) \arrow[r, "\rho"] \arrow[d, "{[f\circ\sigma_1]\cdot}"] & H^1(\mathcal{D}, \mathcal{X}_{\tilde{\mathcal{F}}}) \arrow[r, "\psi"] \arrow[d, "{[\varphi]\cdot}"] & H^1(D', \mathcal{X}_{\tilde{\mathcal{F}}}) \arrow[r] \arrow[d, "{[f\circ\sigma_2\circ\ldots\circ\sigma_\ell]\cdot}"] & 0 \\
0 \arrow[r] & H^1(\tilde{D}, \mathcal{X}_{\tilde{\mathcal{F}}^1}) \arrow[r, "\rho"] & H^1(\mathcal{D}, \mathcal{X}_{\tilde{\mathcal{F}}}) \arrow[r, "\psi"] & H^1(D', \mathcal{X}_{\tilde{\mathcal{F}}}) \arrow[r] & 0
\end{tikzcd}$$

we get $[\varphi]\cdot$ is identically zero, because $[f \circ \sigma_1]\cdot$ and $[f \circ \sigma_2 \circ \ldots \circ \sigma_\ell]\cdot$ are morphisms identically zero by the first step of the proof and induction hypothesis, respectively. □

Now, we prove our main result.

Theorem A. *Let $\mathcal{F}$ be a germ of a second type holomorphic foliation at $(\mathbb{C}^2, p)$ induced by $\omega = P(x, y)dx + Q(x, y)dy$, where $P, Q \in \mathbb{C}\{x, y\}$, and let $F = f/h$ be a reduced balanced equation of separatrices for $\mathcal{F}$. Then f^2 belongs to the ideal (P, Q).*

Proof. Let $\pi : (\tilde{X}, \mathcal{D}) \to (\mathbb{C}^2, p)$ be a reduction of singularities of $\mathcal{F}$ and $\tilde{\mathcal{F}} = \pi^*(\mathcal{F})$ be the strict transform of $\mathcal{F}$ by π. According to Genzmer [6, Proposition 3.1], since $\mathcal{F}$ is of second type, we have the exact sequence of sheaves

$$0 \longrightarrow \mathcal{X}_{\tilde{\mathcal{F}}} \longrightarrow \mathcal{X}_{Z_0} \xrightarrow{\pi^*(\frac{\omega}{F})} \mathcal{O}(-Z_\infty) \longrightarrow 0,$$

where $\mathcal{X}_{\tilde{\mathcal{F}}}$ be the sheaf of vector fields tangent to $\tilde{\mathcal{F}}$ and let $\mathcal{X}_{Z_0}$ be the sheaf of vector fields tangent to the divisor $\mathcal{D}$ and to Z_0. Then, there exists a covering of $\mathcal{D}$ by open subsets $V_i \subset \tilde{X}$ and holomorphic vector fields $X_i \in \mathcal{X}_{Z_0}(V_i)$ such that

$$\pi^*\left(\frac{\omega}{F}\right)(X_i) = h \circ \pi, \quad \mathcal{O}(-Z_\infty) = (h \circ \pi)\mathcal{O},$$

which implies that

$$\pi^*(\omega)(X_i) = (F \circ \pi) \cdot (h \circ \pi) = f \circ \pi. \tag{10}$$

Let $X_{ij} := X_i - X_j$. It follows from (10) that $\pi^*(\omega)(X_{ij}) = 0$. Hence X_{ij} is a 1-cocycle with values over the sheaf $\mathcal{X}_{\tilde{\mathcal{F}}}$ and therefore

$$[(f \circ \pi)X_{ij}] = 0 \in H^1(\mathcal{D}, \mathcal{X}_{\tilde{\mathcal{F}}}),$$

by Proposition 3.1. Thus, there exists a holomorphic vector field $\tilde{v}$ on $\mathcal{D}$ such that $\tilde{v}|_{V_i} = (f \circ \pi) \cdot X_i$. Up multiplication by $f \circ \pi$ in (10), we get

$$\pi^*(\omega)(\tilde{v}) = (f \circ \pi)^2 = f^2 \circ \pi.$$

The direct image of $\tilde{v}$ by π over $(\mathbb{C}^2, p)$ is a holomorphic vector field outside the origin of $\mathbb{C}^2$. The proof ends, by applying Hartogs extension theorem. $\square$

We note that Theorem A is optimal, in the sense, that the hypothesis on the foliation be of second type cannot be removed. For instance, we have the following example.

Example 3.2. Let $\omega = y(2x^8+2(\lambda+1)x^2y^3-y^4)dx+x(y^4-(\lambda+1)x^2y^3-x^8)dy$ be a 1-form defining a singular foliation $\mathcal{F}$ at $(\mathbb{C}^2, 0)$, which is not of second type and $xy = 0$ is the equation of an effective divisor of separatrices for $\mathcal{F}$ (see [5, Example 6.5]). We claim that $(xy)^2$ does not belong to the ideal generated by the coefficients of ω. In fact, if $P(x, y) := y(2x^8+2(\lambda+1)x^2y^3-y^4)$, $Q(x, y) := x(y^4-(\lambda+1)x^2y^3-x^8)$ and we suppose that $(xy)^2 = a(x, y)P(x, y) + b(x, y)Q(x, y)$ for some $a(x, y), b(x, y) \in \mathbb{C}[[x, y]]$ then $4 = \mathrm{ord}(xy)^2 \geq \min\{\mathrm{ord}(a(x, y)P(x, y)), \mathrm{ord}(b(x, y)Q(x, y))\} \geq 5$ which is a contradiction.

The following corollary will be useful in the following section:

Corollary 3.3. *Let $\mathcal{F}$ be a germ of a second type holomorphic foliation at $(\mathbb{C}^2, p)$ induced by $\omega = P(x, y)dx + Q(x, y)dy$, where $P, Q \in \mathbb{C}\{x, y\}$, and let $\mathcal{B}$ be a reduced balanced equation of separatrices for $\mathcal{F}$. If $\mathcal{B}_0 : f(x, y) = 0$ and $\bar{f}$ is the coset of f modulo (P, Q) then the complex vector spaces $(f, P, Q)/(P, Q)$ and $(\bar{f})/(\bar{f}^2)$ are isomorphic.*

Proof. Put $\mathfrak{T} = (f, P, Q)$. The map $\psi : \mathfrak{T} \longrightarrow (\bar{f})/(\bar{f}^2)$ given by

$$\psi(g_z f + \alpha P + \beta Q) = \overline{g_z f} \mod (\bar{f}^2)$$

is an epimorphism of complex vector spaces. Finally by Theorem A the kernel of ψ equals (P, Q). $\square$

4. Milnor and Tjurina numbers after the Briançon-Skoda theorem

Let $\mathcal{F}$ be a singular holomorphic foliation at $(\mathbb{C}^2, p)$ given by the 1-form $\omega := P(x, y)dx + Q(x, y)dy$. Assume that $\mathcal{F}$ has an isolated singularity at p and consider the jacobian ideal associated with $\mathcal{F}$ given by $J(\mathcal{F}) = (P, Q)$. Then $\mathcal{M}(\mathcal{F}) := \mathbb{C}[[x, y]]/J(\mathcal{F})$ is a finite $\mathbb{C}$-dimensional vector space which dimension is called the *Milnor number* of $\mathcal{F}$ and we denote it by $\mu_p(\mathcal{F})$. It is well-known, after [3], that the Milnor number is a topological invariant of the foliation. Let $C : f(x, y) = 0$ be an $\mathcal{F}$-invariant reduced curve. Put $\mathcal{T}(\mathcal{F}, C) := \mathbb{C}[[x, y]]/(f, P, Q)$, where $(\cdot, \cdot, \cdot)$ denotes the ideal generated by three elements in $\mathbb{C}[[x, y]]$.

The *Tjurina number of $\mathcal{F}$ with respect to C* is

$$\tau_p(\mathcal{F}, C) = \dim_{\mathbb{C}} \mathcal{T}(\mathcal{F}, C).$$

Let $\mathcal{B}$ be a balanced divisor of separatrices for $\mathcal{F}$. Put $\mathcal{B}_0 : f(x, y) = 0$ the zero divisor of $\mathcal{B}$. By definition $\tau_p(\mathcal{F}, \mathcal{B}_0) \leq \mu_p(\mathcal{F})$. Put $\mathfrak{T} = (f, P, Q)$. From the third isomorphic theorem for complex vector spaces we have

$$\tau_p(\mathcal{F}, \mathcal{B}_0) = \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/\mathfrak{T} = \dim_{\mathbb{C}} \mathcal{M}(\mathcal{F}) - \dim_{\mathbb{C}} \mathfrak{T}/J(\mathcal{F}),$$

so

$$\mu_p(\mathcal{F}) - \tau_p(\mathcal{F}, \mathcal{B}_0) = \dim_{\mathbb{C}} \mathfrak{T}/J(\mathcal{F}). \tag{11}$$

For any $z \in \mathbb{C}[[x, y]]$ we denote by $\bar{z}$ the coset of z modulo $J(\mathcal{F})$ and $\hat{z}$ its coset modulo $\mathfrak{T}$. Inspired by Liu [9] we consider the exact sequence

$$0 \longrightarrow \operatorname{Ker} \sigma \xrightarrow{i} \mathcal{M}(\mathcal{F}) \xrightarrow{\sigma} \mathcal{M}(\mathcal{F}) \xrightarrow{\delta_{\mathcal{B}}} \mathcal{T}(\mathcal{F}, \mathcal{B}_0) \longrightarrow 0,$$

where i is the inclusion map, σ is the multiplication by $\bar{f}$, that is, $\sigma(\bar{z}) = \overline{zf}$ and $\delta_{\mathcal{B}}(\bar{z}) = \hat{z}$. Since $\delta_{\mathcal{B}}$ is surjective, we get

$$\mu_p(\mathcal{F}) - \tau_p(\mathcal{F}, \mathcal{B}_0) = \dim_{\mathbb{C}} \operatorname{Ker} \delta_{\mathcal{B}}. \tag{12}$$

From (12) and the equality $\mu_p(\mathcal{F}) = \dim_{\mathbb{C}} \operatorname{Ker} \sigma + \dim_{\mathbb{C}} \operatorname{Im} \sigma$, we conclude

$$\tau_p(\mathcal{F}, \mathcal{B}_0) = \dim_{\mathbb{C}} \operatorname{Ker} \sigma = \dim_{\mathbb{C}}(J(\mathcal{F}) : \mathcal{B}_0)/J(\mathcal{F}), \tag{13}$$

where $(J(\mathcal{F}) : \mathcal{B}_0) = \{z \in \mathbb{C}[[x, y]] \ : \ zf \in J(\mathcal{F})\}$.

Proposition 4.1. *Let $\mathcal{F}$ be a singular holomorphic foliation of second type at $(\mathbb{C}^2, p)$ given by the 1-form $\omega = P(x, y)dx + Q(x, y)dy = 0$. Let $\mathcal{B}$ be a balanced divisor of separatrices for $\mathcal{F}$ with $\mathcal{B}_0 : f(x, y) = 0$. Then $\tau_p(\mathcal{F}, \mathcal{B}_0) \leq \mu_p(\mathcal{F}) \leq 2\tau_p(\mathcal{F}, \mathcal{B}_0)$. Moreover $\mu_p(\mathcal{F}) = 2\tau_p(\mathcal{F}, \mathcal{B}_0)$ if and only if $\ker \sigma = (\bar{f})$, where $\bar{f}$ is the coset of f modulo (P, Q).*

Proof. Let us prove the inequality $\mu_p(\mathcal{F}) \leq 2\tau_p(\mathcal{F}, \mathcal{B}_0)$. By Theorem A we get $f^2 \in J(\mathcal{F})$, that is, $\overline{f^2} = \bar{0} \in \mathcal{M}(\mathcal{F})$. Hence, we get the inclusion of ideals $\mathfrak{T} \subseteq \operatorname{Ker} \sigma$. Moreover we have the following chain of ideals of $\mathcal{M}(\mathcal{F})$:

$$\mathcal{M}(\mathcal{F}) \supseteq (\bar{f}) \supseteq (\bar{f^2}) = (\bar{0})$$

where $(\cdot)$ denotes a principal ideal. We also have the exact sequence:

$$0 \to \operatorname{Ker} \sigma \cap (\bar{f}) \xrightarrow{i} (\bar{f}) \xrightarrow{\sigma'} (\bar{f}) \xrightarrow{e} (\bar{f})/(\bar{f^2}) \to 0,$$

where i is the inclusion map, σ' is the multiplication by the coset $\bar{f}$ and e is the natural epimorphism. We have

$$\begin{aligned}\dim_{\mathbb{C}} \operatorname{Ker} \sigma' + \dim_{\mathbb{C}} \operatorname{Im} \sigma' &= \dim_{\mathbb{C}}(\bar{f}) = \dim_{\mathbb{C}} \operatorname{Ker} e + \dim_{\mathbb{C}} \operatorname{Im} e \\ &= \dim_{\mathbb{C}} \operatorname{Im} \sigma' + \dim_{\mathbb{C}}(\bar{f})/(\bar{f^2}),\end{aligned}$$

so from (13) we get

$$\dim_{\mathbb{C}}(\bar{f})/(\bar{f^2}) = \dim_{\mathbb{C}} \operatorname{Ker} \sigma' = \dim_{\mathbb{C}} \operatorname{Ker} \sigma \cap (\bar{f}) \leq \dim_{\mathbb{C}} \operatorname{Ker} \sigma = \tau(\mathcal{F}, \mathcal{B}_0).$$

After Corollary 3.3 we have $\dim_{\mathbb{C}}(\bar{f})/(\bar{f^2}) = \dim_{\mathbb{C}} \mathfrak{T}/J(\mathcal{F})$ and by (11) we conclude $\mu_p(\mathcal{F}) \leq 2\tau_p(\mathcal{F}, \mathcal{B}_0)$. Finally $\mu_p(\mathcal{F}) = 2\tau_p(\mathcal{F}, \mathcal{B}_0)$ if and only if $\operatorname{Ker} \sigma \cap (\bar{f}) = \operatorname{Ker} \sigma$, so $\operatorname{Ker} \sigma \subseteq (\bar{f})$. We conclude the proof since $\sigma(\bar{f}) = \bar{0}$. $\square$

The *intersection multiplicity* of two curves $C : f(x, y) = 0$ and $D : g(x, y) = 0$ at the point p is by definition $i_p(C, D) = \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/(f, g)$ where (f, g) denotes the ideal of $\mathbb{C}\{x, y\}$ generated by the power series f and g.

The *polar curve* of the singular foliation $\mathcal{F} : \omega = P(x, y)dx + Q(x, y)dy = 0$ at $(\mathbb{C}^2, p)$ with respect to a point $(a : b)$ of the complex projective line $\mathbb{P}^1(\mathbb{C})$ is the analytic curve $\mathcal{P}^{\mathcal{F}}_{(a:b)} : aP(x, y) + bQ(x, y) = 0$. There exists an open Zariski set U of $\mathbb{P}^1(\mathbb{C})$ such that $\{aP(x, y) + bQ(x, y) = 0 \ : \ (a : b) \in U\}$ is an equisingular family of plane curves. Any element of this set is called *generic polar curve* of the foliation $\mathcal{F}$ and we will denote it by $\mathcal{P}^{\mathcal{F}}$.

A germ of plane curve $C : f(x, y) = 0$ of multiplicity n is a *semi-homogeneous function* at p if and only if $f = f_n + g$ where f_n is a homogeneous polynomial of degree n defining an isolated singularity at p and g consists of terms of degree at least $n + 1$.

Theorem B. *Let $\mathcal{F}$ be a singular holomorphic foliation of second type at $(\mathbb{C}^2, p)$. Let $\mathcal{B} = \mathcal{B}_0 - \mathcal{B}_\infty$ be a balanced divisor of separatrices for $\mathcal{F}$. Then*

$$\frac{(\nu_p(\mathcal{B}_0) - 1)^2 + \nu_p(\mathcal{B}_\infty) - i_p(\mathcal{P}^{\mathcal{F}}, \mathcal{B}_\infty) - i_p(\mathcal{B}_0, \mathcal{B}_\infty)}{2} \overset{(*)}{\leq} \frac{\mu_p(\mathcal{F})}{2} \leq \tau_p(\mathcal{F}, \mathcal{B}_0), \tag{14}$$

and the equality $(*)$ holds if $\mathcal{F}$ is a generalized curve foliation and $\mathcal{B}_0$ is defined by a germ of semi-homogeneous function at p. Moreover, if $\mathcal{B}_\infty = \emptyset$, then

$$\frac{\nu_p(\mathcal{F})^2}{2} \leq \frac{\mu_p(\mathcal{F})}{2} \leq \tau_p(\mathcal{F}, \mathcal{B}_0).$$

Proof. By [5, Proposition 4.2], for any singular foliation $\mathcal{F}$ we have

$$\Delta_p(\mathcal{F}, \mathcal{B}_0) = i_p(\mathcal{P}^{\mathcal{F}}, \mathcal{B}_0) + i_p(\mathcal{B}_0, \mathcal{B}_\infty) - \mu_p(\mathcal{B}_0) - \nu_p(\mathcal{B}_0) + 1, \tag{15}$$

where $\Delta_p(\mathcal{F}, \mathcal{B}_0)$ is the excess polar number of $\mathcal{F}$ with respect to $\mathcal{B}_0$. Since $\mathcal{F}$ is of second type, $\nu_p(\mathcal{F}) = \nu_p(\mathcal{B}) - 1 = \nu_p(\mathcal{B}_0) - \nu_p(\mathcal{B}_\infty) - 1$ by Eq. (8), and therefore, from (15) we get

$$\Delta_p(\mathcal{F}, \mathcal{B}_0) = i_p(\mathcal{P}^{\mathcal{F}}, \mathcal{B}_0) + i_p(\mathcal{B}_0, \mathcal{B}_\infty) - \mu_p(\mathcal{B}_0) - \nu_p(\mathcal{F}) - \nu_p(\mathcal{B}_\infty). \tag{16}$$

On the other hand, after [7, Theorem A] we know that $\Delta_p(\mathcal{F}, \mathcal{B}_0) \geq 0$, and equals zero if and only if $\mathcal{F}$ is a generalized curve foliation. Hence from (16) we have

$$\mu_p(\mathcal{B}_0) \leq i_p(\mathcal{P}^{\mathcal{F}}, \mathcal{B}_0) + i_p(\mathcal{B}_0, \mathcal{B}_\infty) - \nu_p(\mathcal{F}) - \nu_p(\mathcal{B}_\infty). \tag{17}$$

Now, by applying [5, Lemma 4.4] to $\mathcal{F}$, which is of second type, and by properties of the intersection multiplicity one gets

$$i_p(\mathcal{P}^{\mathcal{F}}, \mathcal{B}_0) = i_p(\mathcal{P}^{\mathcal{F}}, \mathcal{B}_\infty) + \mu_p(\mathcal{F}) + \nu_p(\mathcal{F}), \tag{18}$$

so from (17) and (18),

$$\mu_p(\mathcal{B}_0) \leq \mu_p(\mathcal{F}) + i_p(\mathcal{B}_0, \mathcal{B}_\infty) + i_p(\mathcal{P}^{\mathcal{F}}, \mathcal{B}_\infty) - \nu_p(\mathcal{B}_\infty). \tag{19}$$

It follows from the definition of the Milnor number, the properties of the intersection multiplicity and (19) that

$$(\nu_p(\mathcal{B}_0) - 1)^2 \leq \mu_p(\mathcal{B}_0) \leq \mu_p(\mathcal{F}) + i_p(\mathcal{B}_0, \mathcal{B}_\infty) + i_p(\mathcal{P}^{\mathcal{F}}, \mathcal{B}_\infty) - \nu_p(\mathcal{B}_\infty). \tag{20}$$

Observe that the first inequality becomes an equality when B_0 is defined by a germ of semi-homogeneous function at p (see [17]) and the second inequality is an equality if and only if $\mathcal{F}$ is a generalized curve foliation. Finally, the proof ends, up applying Proposition 4.1

$$(\nu_p(\mathcal{B}_0) - 1)^2 + \nu_p(\mathcal{B}_\infty) - i_p(\mathcal{B}_0, \mathcal{B}_\infty) - i_p(\mathcal{P}^{\mathcal{F}}, \mathcal{B}_\infty) \leq \mu_p(\mathcal{F}) \leq 2\tau_p(\mathcal{F}, \mathcal{B}_0). \quad \square \tag{21}$$

Example 4.2. We illustrate Theorem B with the radial foliation $\mathcal{F}$ given by the 1-form $\omega = xdy - ydx$. In this case we consider $\mathcal{B}_0 = xy(x-y)$ and $\mathcal{B}_\infty = x + y$. We get $\nu_0(\mathcal{B}_0) = 3$, $1 = \nu_0(\mathcal{B}_\infty) = i_0(\mathcal{P}^{\mathcal{F}}, \mathcal{B}_\infty) = \tau_0(\mathcal{F}, \mathcal{B}_0)$ and $i_0(\mathcal{B}_0, \mathcal{B}_\infty) = 3$. Hence $\mathcal{F}$ verifies (14).

Remark 4.3. The family of foliations given in [5, Example 6.5] are defined by the 1-form

$$\omega_k = y(2x^{2k-2} + 2(\lambda+1)x^2y^{k-2} - y^{k-1})dx + x(y^{k-1} - (\lambda+1)x^2y^{k-2} - x^{2k-2})dy$$

is a family of dicritical foliations which are not of second type, $\mathcal{B} = (x) + (y)$ is an effective balanced divisor of separatrices for $\mathcal{F}_k$. We get $\nu_0(\mathcal{F}_k) = k$ and $\tau_0(\mathcal{F}_k, \mathcal{B}) = 3k - 2$. Hence the inequality

$$\frac{\nu_p(\mathcal{F})^2}{2} \leq \tau_p(\mathcal{F}, \mathcal{B})$$

fails for all $k \geq 6$. Therefore, in Theorem B the second type hypothesis over $\mathcal{F}$ is essential.

5. A lower bound for the global Tjurina number of an algebraic curve

Let C be a reduced curve of degree $\deg(C)$ in the complex projective plane $\mathbb{P}^2$. Denote by $\tau(C)$ the *global Tjurina number* of the curve C, which is the sum of the Tjurina numbers at the singular points of C. In this section, under some conditions, we give a lower bound for $\tau(C)$.

A holomorphic foliation $\mathcal{F}$ on $\mathbb{P}^2$ of degree $d \geq 0$ is a foliation defined by a polynomial 1-form $\Omega = A(x, y, z)dx + B(x, y, z)dy + C(x, y, z)dz$, where A, B, C are complex homogeneous polynomials of degree $d + 1$, satisfying two conditions:

(1) the integrability condition $\Omega \wedge d\Omega = 0$,
(2) the Euler condition $Ax + By + Cz = 0$.

An algebraic curve $C : f(x, y, z) = 0$ is $\mathcal{F}$-invariant if $\Omega \wedge df = f\Theta$, where Θ is some polynomial 2-form.

Denote by $\lceil z \rceil$ the ceiling function evaluated at $z \in \mathbb{R}$, that is, the smallest integer that is greater than or equal to $z \in \mathbb{R}$. We have:

Theorem 5.1. *Let $\mathcal{F}$ be a holomorphic foliation on $\mathbb{P}^2$ of degree d. Suppose that all points $p \in Sing(\mathcal{F})$ are of second type. Then*

$$\left\lceil \frac{d^2 + d + 1 - 2\sum_{p\in Sing(\mathcal{F})} GSV_p(\mathcal{F}, (F_p)_0)}{2} \right\rceil \leq \sum_{p\in Sing(\mathcal{F})} \tau_p((F_p)_0), \tag{22}$$

where $(F_p)_0$ is the zero divisor of a balanced equation of separatrices F_p for $\mathcal{F}$ at p. In particular, if C is an $\mathcal{F}$-invariant reduced curve in $\mathbb{P}^2$ such that $Sing(\mathcal{F}) \subset C$ and for all $p \in Sing(\mathcal{F})$, the germ of C at p defines the zero divisor of a balanced equation of separatrices for $\mathcal{F}$ at p, then

$$\left\lceil \frac{d^2 + d + 1 - 2(d + 2)\deg(C) + 2\deg(C)^2}{2} \right\rceil \leq \tau(C), \tag{23}$$

Proof. Since all points $p \in Sing(\mathcal{F})$ are of second type, then

$$\mu_p(\mathcal{F}) \leq 2\tau_p(\mathcal{F}, (F_p)_0) \tag{24}$$

by Theorem B. According to [5, Proposition 6.2], we have $\tau_p(\mathcal{F}, (F_p)_0) = GSV_p(\mathcal{F}, (F_p)_0) + \tau_p((F_p)_0)$. Hence, up substituting in (24), we obtain

$$\frac{\mu_p(\mathcal{F}) - 2GSV_p(\mathcal{F}, (F_p)_0)}{2} \leq \tau_p((F_p)_0), \quad \text{for all } p \in Sing(\mathcal{F}).$$

The inequality (22) is proved by taking sum over all singular points of $\mathcal{F}$, by using $\sum_{p\in Sing(\mathcal{F})} \mu_p(\mathcal{F}) = d^2 + d + 1$ (see [2, Page 19]) and considering the ceiling function. The inequality (23) follows from

$$\sum_{p\in Sing(\mathcal{F})\cap C} GSV_p(\mathcal{F}, C) = (d + 2)\deg(C) - \deg(C)^2$$

given in [1, Proposition 4] and considering again the ceiling function. □

The following example illustrates Theorem 5.1.

Example 5.2. For each $\lambda \in \mathbb{C}$, we consider the 1-form

$$\omega_\lambda = yzdx + \lambda xzdy - (\lambda + 1)xydz,$$

which defines a foliation $\mathcal{F}_\lambda$ on $\mathbb{P}^2$ of degree one. The curve $C : xyz = 0$ has degree three and it satisfies all hypotheses of Theorem 5.1. Then

$$\left\lceil \frac{1^2 + 1 + 1 - 2(1+2)3 + 2 \cdot 3^2}{2} \right\rceil = \left\lceil \frac{3}{2} \right\rceil = 2 \leq \tau(C) = 3,$$

which implies that the inequality (23) of Theorem 5.1 is verified. Observed that we equate the bound given by du Plessis and Wall in [4, Theorem 3.2].

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Data availability

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